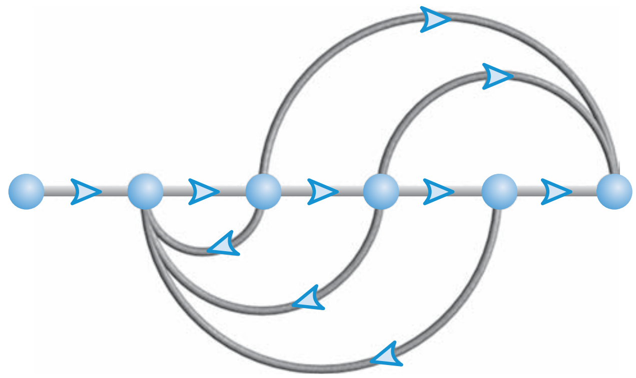
Time Response

4



Chapter Learning Outcomes

After completing this chapter the student will be able to:

- Use poles and zeros of transfer functions to determine the time response of a control system (Sections 4.1–4.2)
- Describe quantitatively the transient response of first-order systems (Section 4.3)
- Write the general response of second-order systems given the pole location (Section 4.4)
- Find the damping ratio and natural frequency of a second-order system (Section 4.5)
- Find the settling time, peak time, percent overshoot, and rise time for an underdamped second-order system (Section 4.6)
- Approximate higher-order systems and systems with zeros as first- or second-order systems (Sections 4.7–4.8)
- Describe the effects of nonlinearities on the system time response (Section 4.9)
- Find the time response from the state-space representation (Sections 4.10–4.11)

Case Study Learning Outcomes

You will be able to demonstrate your knowledge of the chapter objectives with case studies as follows:

- Given the antenna azimuth position control system shown on the front endpapers, you will be able to (1) predict, by inspection, the form of the open-loop angular velocity response of the load to a step voltage input to the power amplifier; (2) describe quantitatively the transient response of the open-loop system; (3) derive the expression for the open-loop angular velocity output for a step voltage input; (4) obtain the open-loop state-space representation; (5) plot the open-loop velocity step response using a computer simulation.

- Given the block diagram for the Unmanned Free-Swimming Submersible (UFSS) vehicle's pitch control system shown on the back endpapers, you will be able to predict, find, and plot the response of the vehicle dynamics to a step input command. Further, you will be able to evaluate the effect of system zeros and higher-order poles on the response. You also will be able to evaluate the roll response of a ship at sea.

4.1 Introduction

In Chapter 2, we saw how transfer functions can represent linear, time-invariant systems. In Chapter 3, systems were represented directly in the time domain via the state and output equations. After the engineer obtains a mathematical representation of a subsystem, the subsystem is analyzed for its transient and steady-state responses to see if these characteristics yield the desired behavior. This chapter is devoted to the analysis of system transient response.

It may appear more logical to continue with Chapter 5, which covers the modeling of closed-loop systems, rather than to break the modeling sequence with the analysis presented here in Chapter 4. However, the student should not continue too far into system representation without knowing the application for the effort expended. Thus, this chapter demonstrates applications of the system representation by evaluating the transient response from the system model. Logically, this approach is not far from reality, since the engineer may indeed want to evaluate the response of a subsystem prior to inserting it into the closed-loop system.

After describing a valuable analysis and design tool, poles and zeros, we begin analyzing our models to find the step response of first- and second-order systems. The order refers to the order of the equivalent differential equation representing the system—the order of the denominator of the transfer function after cancellation of common factors in the numerator or the number of simultaneous first-order equations required for the state-space representation.

4.2 Poles, Zeros, and System Response

The output response of a system is the sum of two responses: the *forced response* and the *natural response*.[1] Although many techniques, such as solving a differential equation or taking the inverse Laplace transform, enable us to evaluate this output response, these techniques are laborious and time-consuming. Productivity is aided by analysis and design techniques that yield results in a minimum of time. If the technique is so rapid that we feel we derive the desired result by inspection, we sometimes use the attribute *qualitative* to describe the method. The use of poles and zeros and their relationship to the time response of a system is such a technique. Learning this relationship gives us a qualitative "handle" on problems. The concept of poles and zeros, fundamental to the analysis and design of control

[1] The forced response is also called the *steady-state response* or *particular solution*. The natural response is also called the *homogeneous solution*.

systems, simplifies the evaluation of a system's response. The reader is encouraged to master the concepts of poles and zeros and their application to problems throughout this book. Let us begin with two definitions.

Poles of a Transfer Function

The *poles* of a transfer function are (1) the values of the Laplace transform variable, s, that cause the transfer function to become infinite or (2) any roots of the denominator of the transfer function that are common to roots of the numerator.

Strictly speaking, the poles of a transfer function satisfy part (1) of the definition. For example, the roots of the characteristic polynomial in the denominator are values of s that make the transfer function infinite, so they are thus poles. However, if a factor of the denominator can be canceled by the same factor in the numerator, the root of this factor no longer causes the transfer function to become infinite. In control systems, we often refer to the root of the canceled factor in the denominator as a pole even though the transfer function will not be infinite at this value. Hence, we include part (2) of the definition.

Zeros of a Transfer Function

The *zeros* of a transfer function are (1) the values of the Laplace transform variable, s, that cause the transfer function to become zero, or (2) any roots of the numerator of the transfer function that are common to roots of the denominator.

Strictly speaking, the zeros of a transfer function satisfy part (1) of this definition. For example, the roots of the numerator are values of s that make the transfer function zero and are thus zeros. However, if a factor of the numerator can be canceled by the same factor in the denominator, the root of this factor no longer causes the transfer function to become zero. In control systems, we often refer to the root of the canceled factor in the numerator as a zero even though the transfer function will not be zero at this value. Hence, we include part (2) of the definition.

Poles and Zeros of a First-Order System: An Example

Given the transfer function $G(s)$ in Figure 4.1(a), a pole exists at $s = -5$, and a zero exists at -2. These values are plotted on the complex s-plane in Figure 4.1(b), using an $\times$ for the pole and a $\bigcirc$ for the zero. To show the properties of the poles and zeros, let us find the unit step response of the system. Multiplying the transfer function of Figure 4.1(a) by a step function yields

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5} \tag{4.1}$$

where

$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} = \frac{2}{5}$$

$$B = \left.\frac{(s+2)}{s}\right|_{s\to -5} = \frac{3}{5}$$

Thus,

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t} \tag{4.2}$$

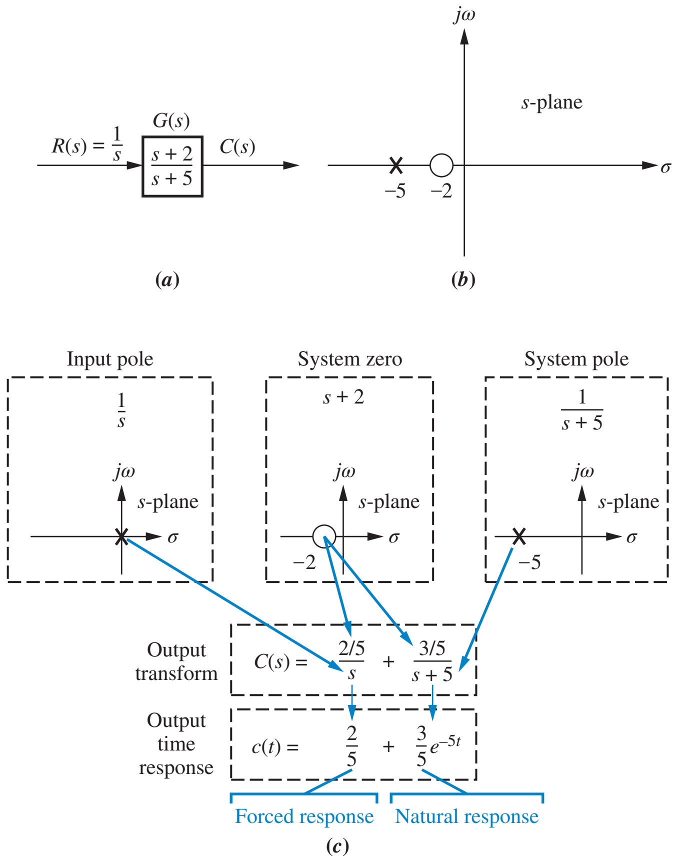


FIGURE 4.1 **a.** System showing input and output; **b.** pole-zero plot of the system; **c.** evolution of a system response. Follow blue arrows to see the evolution of the response component generated by the pole or zero.

From the development summarized in Figure 4.1(c), we draw the following conclusions:

1. A pole of the input function generates the form of the *forced response* (that is, the pole at the origin generated a step function at the output).
2. A pole of the transfer function generates the form of the *natural response* (that is, the pole at -5 generated e^{-5t}).
3. A pole on the real axis generates an *exponential* response of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis. Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero (again, the pole at -5 generated e^{-5t}; see Figure 4.2 for the general case).

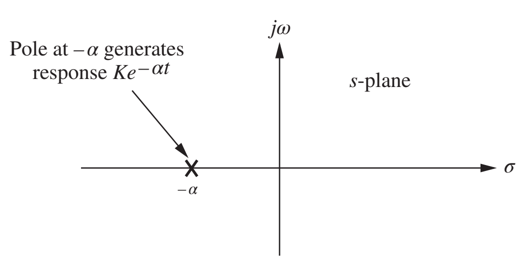


FIGURE 4.2 Effect of a real-axis pole upon transient response.

4. The zeros and poles generate the *amplitudes* for both the forced and natural responses (this can be seen from the calculation of A and B in Eq. (4.1)).

Let us now look at an example that demonstrates the technique of using poles to obtain the form of the system response. We will learn to write the form of the response by inspection. Each pole of the system transfer function that is on the real axis generates an exponential response that is a component of the natural response. The input pole generates the forced response.

Example 4.1

Evaluating Response Using Poles

PROBLEM: Given the system of Figure 4.3, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

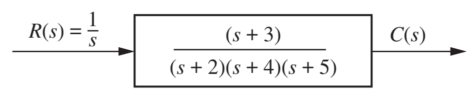


FIGURE 4.3 System for Example 4.1

SOLUTION: By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}} \tag{4.3}$$

Taking the inverse Laplace transform, we get

$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}} \tag{4.4}$$

Skill-Assessment Exercise 4.1

PROBLEM: A system has a transfer function, $G(s) = \dfrac{10(s+4)(s+6)}{(s+1)(s+7)(s+8)(s+10)}$.

Write, by inspection, the output, $c(t)$, in general terms if the input is a unit step.

ANSWER: $c(t) \equiv A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$

In this section, we learned that poles determine the nature of the time response: Poles of the input function determine the form of the forced response, and poles of the transfer function determine the form of the natural response. Zeros and poles of the input or transfer function contribute to the amplitudes of the component parts of the total response. Finally, poles on the real axis generate exponential responses.

4.3 First-Order Systems

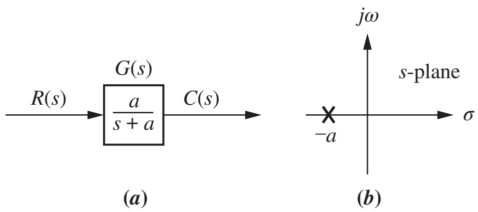


FIGURE 4.4 **a.** First-order system; **b.** pole plot

We now discuss first-order systems without zeros to define a performance specification for such a system. A first-order system without zeros can be described by the transfer function shown in Figure 4.4(a). If the input is a unit step, where $R(s) = 1/s$, the Laplace transform of the step response is $C(s)$, where

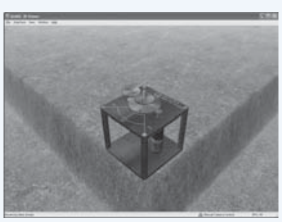
$$C(s) = R(s)G(s) = \frac{a}{s(s+a)} \tag{4.5}$$

Taking the inverse transform, the step response is given by

$$\boxed{c(t) = c_f(t) + c_n(t) = 1 - e^{-at}} \tag{4.6}$$

Virtual Experiment 4.1
First-Order Transfer Function

Put theory into practice and find a first-order transfer function representing the Quanser Rotary Servo. Then validate the model by simulating it in LabVIEW. Such a servo motor is used in mechatronic gadgets such as cameras.



Virtual experiments are found on Learning Space.

where the input pole at the origin generated the forced response $c_f(t) = 1$, and the system pole at $-a$, as shown in Figure 4.4(b), generated the natural response $c_n(t) = -e^{-at}$. Equation (4.6) is plotted in Figure 4.5.

Let us examine the significance of parameter a, the only parameter needed to describe the transient response. When $t = 1/a$,

$$e^{-at}\big|_{t=1/a} = e^{-1} = 0.37 \tag{4.7}$$

or

$$c(t)\big|_{t=1/a} = 1 - e^{-at}\big|_{t=1/a} = 1 - 0.37 = 0.63 \tag{4.8}$$

We now use Eqs. (4.6), (4.7), and (4.8) to define three transient response performance specifications.

Time Constant

We call $1/a$ the *time constant* of the response. From Eq. (4.7), the time constant can be described as the time for e^{-at} to decay to 37% of its initial value. Alternately, from Eq. (4.8) the time constant is the time it takes for the step response to rise to 63% of its final value (see Figure 4.5).

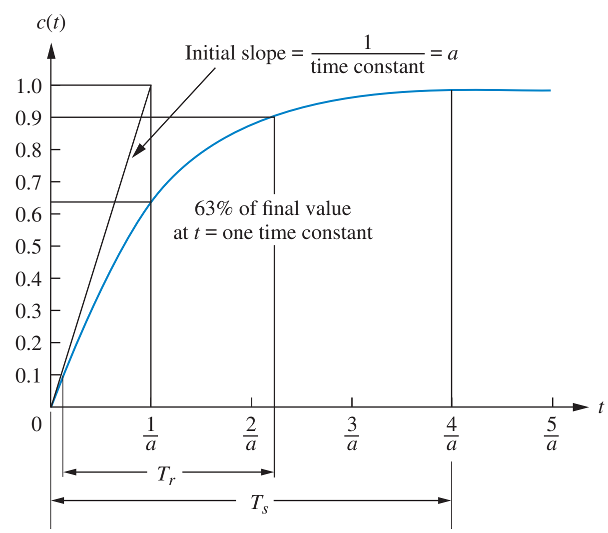


FIGURE 4.5 First-order system response to a unit step

The reciprocal of the time constant has the units (1/seconds), or frequency. Thus, we can call the parameter a the *exponential frequency*. Since the derivative of e^{-at} is $-a$ when $t = 0$, a is the initial rate of change of the exponential at $t = 0$. Thus, the time constant can be considered a transient response specification for a first-order system, since it is related to the speed at which the system responds to a step input.

The time constant can also be evaluated from the pole plot (see Figure 4.4(b)). Since the pole of the transfer function is at $-a$, we can say the pole is located at the *reciprocal* of the time constant, and the farther the pole from the imaginary axis, the faster the transient response.

Let us look at other transient response specifications, such as rise time, T_r, and settling time, T_s, as shown in Figure 4.5.

Rise Time, T_r

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value. Rise time is found by solving Eq. (4.6) for the difference in time at $c(t) = 0.9$ and $c(t) = 0.1$. Hence,

$$\boxed{T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}} \tag{4.9}$$

Settling Time, T_s

Settling time is defined as the time for the response to reach, and stay within, 2% of its final value.[2] Letting $c(t) = 0.98$ in Eq. (4.6) and solving for time, t, we find the settling time to be

$$\boxed{T_s = \frac{4}{a}} \tag{4.10}$$

First-Order Transfer Functions via Testing

Often it is not possible or practical to obtain a system's transfer function analytically. Perhaps the system is closed, and the component parts are not easily identifiable. Since the transfer function is a representation of the system from input to output, the system's step response can lead to a representation even though the inner construction is not known. With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

Consider a simple first-order system, $G(s) = K/(s + a)$, whose step response is

$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)} \tag{4.11}$$

If we can identify K and a from laboratory testing, we can obtain the transfer function of the system.

For example, assume the unit step response given in Figure 4.6. We determine that it has the first-order characteristics we have seen thus far, such as no overshoot and nonzero initial slope. From the response, we measure the time constant, that is, the time for the amplitude to reach 63% of its final value. Since the final value is about 0.72, the time constant is evaluated where the curve reaches $0.63 \times 0.72 = 0.45$, or about 0.13 second. Hence, $a = 1/0.13 = 7.7$.

[2] Strictly speaking, this is the definition of the 2% *setting time*. Other percentages, for example 5%, also can be used. We will use *settling time* throughout the book to mean 2% settling time.

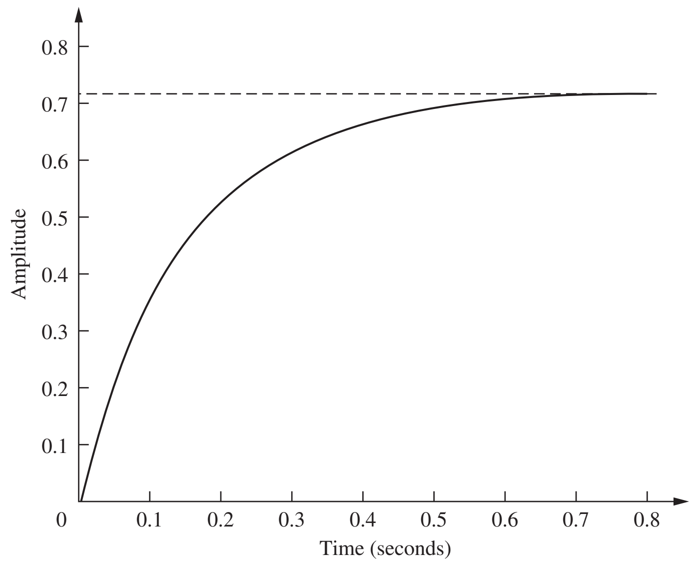


FIGURE 4.6 Laboratory results of a system step response test

To find K, we realize from Eq. (4.11) that the forced response reaches a steady-state value of $K/a = 0.72$. Substituting the value of a, we find $K = 5.54$. Thus, the transfer function for the system is $G(s) = 5.54/(s + 7.7)$. It is interesting to note that the response of Figure 4.6 was generated using the transfer function $G(s) = 5/(s + 7)$.

Skill-Assessment Exercise 4.2

PROBLEM: A system has a transfer function, $G(s) = \frac{50}{s + 50}$. Find the time constant, T_c, settling time, T_s, and rise time, T_r.

ANSWER: $T_c = 0.02 \text{ s}, \; T_s = 0.08 \text{ s}, \text{ and } T_r = 0.044 \text{ s}.$

The complete solution is located at www.wiley.com/college/nise.

4.4 Second-Order Systems: Introduction

Let us now extend the concepts of poles and zeros and transient response to second-order systems. Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described. Whereas varying a first-order system's parameter simply changes the speed of the response, changes in the parameters of a second-order system can change the *form* of the response. For example, a second-order system can display characteristics much like a first-order system, or, depending on component values, display damped or pure oscillations for its transient response.

To become familiar with the wide range of responses before formalizing our discussion in the next section, we take a look at numerical examples of the second-order system responses shown in Figure 4.7. All examples are derived from Figure 4.7(a), the general case, which has two finite poles and no zeros. The term in the numerator is simply a scale or input multiplying factor that can take on any value without affecting the form of the derived results. By assigning appropriate values to parameters a and b, we can show all possible second-order transient responses. The unit step response then can be found using

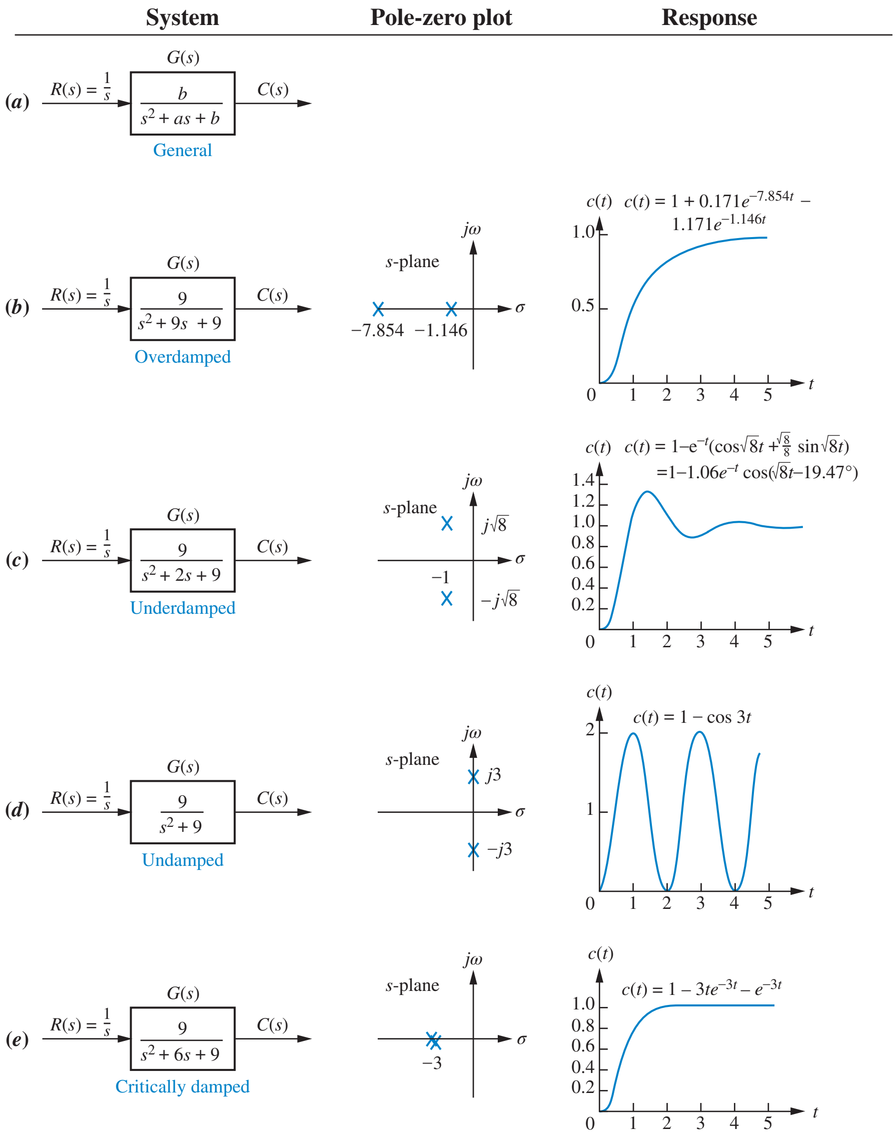


FIGURE 4.7 Second-order systems, pole plots, and step responses

$C(s) = R(s)G(s)$, where $R(s) = 1/s$, followed by a partial-fraction expansion and the inverse Laplace transform. Details are left as an end-of-chapter problem, for which you may want to review Section 2.2.

We now explain each response and show how we can use the poles to determine the nature of the response without going through the procedure of a partial-fraction expansion followed by the inverse Laplace transform.

Overdamped Response, Figure 4.7(*b*)

For this response,

$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)} \tag{4.12}$$

This function has a pole at the origin that comes from the unit step input and two real poles that come from the system. The input pole at the origin generates the constant forced response; each of the two system poles on the real axis generates an exponential natural response whose exponential frequency is equal to the pole location. Hence, the output initially could have been written as $c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$. This response, shown in Figure 4.7(*b*), is called *overdamped*.[3] We see that the poles tell us the form of the response without the tedious calculation of the inverse Laplace transform.

Underdamped Response, Figure 4.7 (*c*)

For this response,

$$C(s) = \frac{9}{s(s^2+2s+9)} \tag{4.13}$$

This function has a pole at the origin that comes from the unit step input and two complex poles that come from the system. We now compare the response of the second-order system to the poles that generated it. First we will compare the pole location to the time function, and then we will compare the pole location to the plot. From Figure 4.7(*c*), the poles that generate the natural response are at $s = -1 \pm j\sqrt{8}$. Comparing these values to $c(t)$ in the same figure, we see that the real part of the pole matches the exponential decay frequency of the sinusoid's amplitude, while the imaginary part of the pole matches the frequency of the sinusoidal oscillation.

Let us now compare the pole location to the plot. Figure 4.8 shows a general, damped sinusoidal response for a second-order system. The transient response consists of an exponentially decaying amplitude generated by the real part of the system pole times a sinusoidal waveform generated by the imaginary part of the system pole. The time constant of the exponential decay is equal to the reciprocal of the real part of the system pole. The value of the imaginary part is the actual frequency of the sinusoid, as depicted in Figure 4.8. This sinusoidal frequency is given the name *damped frequency of oscillation*, ω_d. Finally, the steady-state response (unit step) was generated by the input pole located at the origin. We call the type of response shown in Figure 4.8 an *underdamped response,* one which approaches a steady-state value via a transient response that is a damped oscillation.

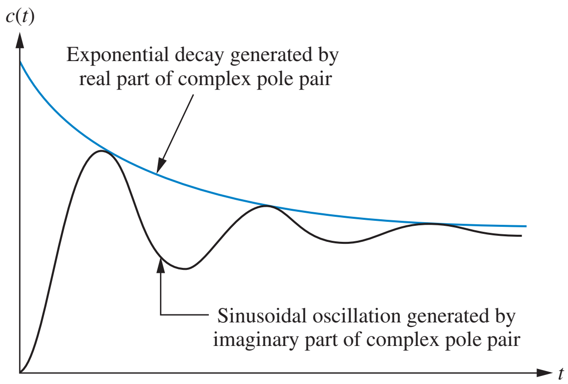


FIGURE 4.8 Second-order step response components generated by complex poles

The following example demonstrates how a knowledge of the relationship between the pole location and the transient response can lead rapidly to the response form without calculating the inverse Laplace transform.

Example 4.2

Form of Underdamped Response Using Poles

PROBLEM: By inspection, write the form of the step response of the system in Figure 4.9.

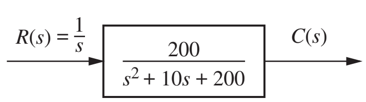


FIGURE 4.9 System for Example 4.2

SOLUTION: First we determine that the form of the forced response is a step. Next we find the form of the natural response. Factoring the denominator of the transfer function in Figure 4.9, we find the poles to be $s = -5 \pm j13.23$. The real part, -5, is the exponential frequency for the damping. It is also the reciprocal of the time constant of the decay of the oscillations. The imaginary part, 13.23, is the radian frequency for the sinusoidal oscillations. Using our previous discussion and Figure 4.7(*c*) as a guide, we obtain $c(t) = K_1 + e^{-5t}(K_2 \cos 13.23t + K_3 \sin 13.23t) = K_1 + K_4 e^{-5t}(\cos 13.23t - \phi)$, where $\phi = \tan^{-1} K_3/K_2$, $K_4 = \sqrt{K_2^2 + K_3^2}$, and $c(t)$ is a constant plus an exponentially damped sinusoid.

[3] So named because *overdamped* refers to a large amount of energy absorption in the system, which inhibits the transient response from overshooting and oscillating about the steady-state value for a step input. As the energy absorption is reduced, an overdamped system will become underdamped and exhibit overshoot.

We will revisit the second-order underdamped response in Sections 4.5 and 4.6, where we generalize the discussion and derive some results that relate the pole position to other parameters of the response.

Undamped Response, Figure 4.7(*d*)

For this response,

$$C(s) = \frac{9}{s(s^2 + 9)} \tag{4.14}$$

This function has a pole at the origin that comes from the unit step input and two imaginary poles that come from the system. The input pole at the origin generates the constant forced response, and the two system poles on the imaginary axis at $\pm j3$ generate a sinusoidal natural response whose frequency is equal to the location of the imaginary poles. Hence, the output can be estimated as $c(t) = K_1 + K_4 \cos(3t - \phi)$. This type of response, shown in Figure 4.7(*d*), is called *undamped*. Note that the absence of a real part in the pole pair corresponds to an exponential that does not decay. Mathematically, the exponential is $e^{-0t} = 1$.

Critically Damped Response, Figure 4.7 (*e*)

For this response,

$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{9}{s(s+3)^2} \tag{4.15}$$

This function has a pole at the origin that comes from the unit step input and two multiple real poles that come from the system. The input pole at the origin generates the constant forced response, and the two poles on the real axis at -3 generate a natural response consisting of an exponential and an exponential multiplied by time, where the exponential frequency is equal to the location of the real poles. Hence, the output can be estimated as $c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$. This type of response, shown in Figure 4.7(*e*), is called *critically damped*. Critically damped responses are the fastest possible without the overshoot that is characteristic of the underdamped response.

We now summarize our observations. In this section we defined the following natural responses and found their characteristics:

1. *Overdamped responses*

   Poles: Two real at $-\sigma_1, \ -\sigma_2$

   Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

   $$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. *Underdamped responses*

   Poles: Two complex at $-\sigma_d \pm j\omega_d$

   Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

   $$c(t) = A e^{-\sigma_d t} \cos(\omega_d t - \phi)$$

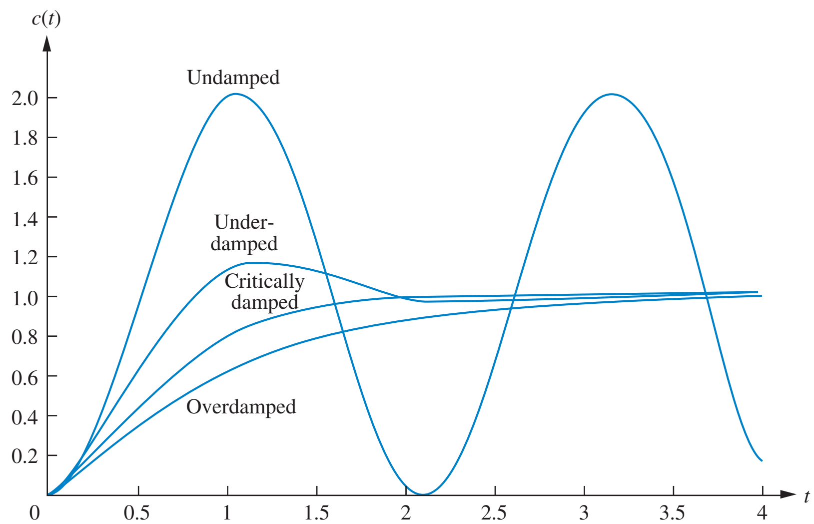


FIGURE 4.10 Step responses for second-order system damping cases

3. *Undamped responses*

Poles: Two imaginary at $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\cos(\omega_1 t - \phi)$$

4. *Critically damped responses*

Poles: Two real at $-\sigma_1$

Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

The step responses for the four cases of damping discussed in this section are superimposed in Figure 4.10. Notice that the critically damped case is the division between the overdamped cases and the underdamped cases and is the fastest response without overshoot.

Skill-Assessment Exercise 4.3

PROBLEM: For each of the following transfer functions, write, by inspection, the general form of the step response:

a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$

b. $G(s) = \dfrac{900}{s^2 + 90s + 900}$

c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$

d. $G(s) = \dfrac{625}{s^2 + 625}$

ANSWERS:

a. $c(t) = A + Be^{-6t}\cos(19.08t + \phi)$

b. $c(t) = A + Be^{-78.54t} + Ce^{-11.46t}$

c. $c(t) = A + Be^{-15t} + Cte^{-15t}$

d. $c(t) = A + B\cos(25t + \phi)$

The complete solution is located at www.wiley.com/college/nise.

In the next section, we will formalize and generalize our discussion of second-order responses and define two specifications used for the analysis and design of second-order systems. In Section 4.6, we will focus on the *underdamped* case and derive some specifications unique to this response that we will use later for analysis and design.

4.5 The General Second-Order System

Now that we have become familiar with second-order systems and their responses, we generalize the discussion and establish quantitative specifications defined in such a way that the response of a second-order system can be described to a designer without the need for sketching the response. In this section, we define two physically meaningful specifications for second-order systems. These quantities can be used to describe the characteristics of the second-order transient response just as time constants describe the first-order system response. The two quantities are called natural frequency and damping ratio. Let us formally define them.

Natural Frequency, ω_n

The *natural frequency* of a second-order system is the frequency of oscillation of the system without damping. For example, the frequency of oscillation of a series *RLC* circuit with the resistance shorted would be the natural frequency.

Damping Ratio, ζ

Before we state our next definition, some explanation is in order. We have already seen that a second-order system's underdamped step response is characterized by damped oscillations. Our definition is derived from the need to quantitatively describe this damped oscillation regardless of the time scale. Thus, a system whose transient response goes through three cycles in a millisecond before reaching the steady state would have the same measure as a system that went through three cycles in a millennium before reaching the steady state. For example, the underdamped curve in Figure 4.10 has an associated measure that defines its shape. This measure remains the same even if we change the time base from seconds to microseconds or to millennia.

A viable definition for this quantity is one that compares the exponential decay frequency of the envelope to the natural frequency. This ratio is constant regardless of the time scale of the response. Also, the reciprocal, which is proportional to the ratio of the natural period to the exponential time constant, remains the same regardless of the time base.

We define the *damping ratio*, ζ, to be

$$\boxed{\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural period (seconds)}}{\text{Exponential time constant}}}$$

Let us now revise our description of the second-order system to reflect the new definitions. The general second-order system shown in Figure 4.7(a) can be transformed to show the quantities ζ and ω_n. Consider the general system

$$G(s) = \frac{b}{s^2 + as + b} \tag{4.16}$$

Without damping, the poles would be on the $j\omega$-axis, and the response would be an undamped sinusoid. For the poles to be purely imaginary, $a = 0$. Hence,

$$G(s) = \frac{b}{s^2 + b} \tag{4.17}$$

By definition, the natural frequency, ω_n, is the frequency of oscillation of this system. Since the poles of this system are on the $j\omega$-axis at $\pm j\sqrt{b}$,

$$\omega_n = \sqrt{b} \tag{4.18}$$

Hence,

$$b = \omega_n^2 \tag{4.19}$$

Now what is the term a in Eq. (4.16)? Assuming an underdamped system, the complex poles have a real part, σ, equal to $-a/2$. The magnitude of this value is then the exponential decay frequency described in Section 4.4. Hence,

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n} \tag{4.20}$$

from which

$$a = 2\zeta\omega_n \tag{4.21}$$

Our general second-order transfer function finally looks like this:

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \tag{4.22}$$

In the following example we find numerical values for ζ and ω_n by matching the transfer function to Eq. (4.22).

Example 4.3

Finding ζ and ω_n For a Second-Order System

PROBLEM: Given the transfer function of Eq. (4.23), find ζ and ω_n.

$$G(s) = \frac{36}{s^2 + 4.2s + 36} \tag{4.23}$$

SOLUTION: Comparing Eq. (4.23) to (4.22), $\omega_n^2 = 36$, from which $\omega_n = 6$. Also, $2\zeta\omega_n = 4.2$. Substituting the value of ω_n, $\zeta = 0.35$.

Now that we have defined ζ and ω_n, let us relate these quantities to the pole location. Solving for the poles of the transfer function in Eq. (4.22) yields

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{4.24}$$

From Eq. (4.24) we see that the various cases of second-order response are a function of ζ; they are summarized in Figure 4.11.[4]

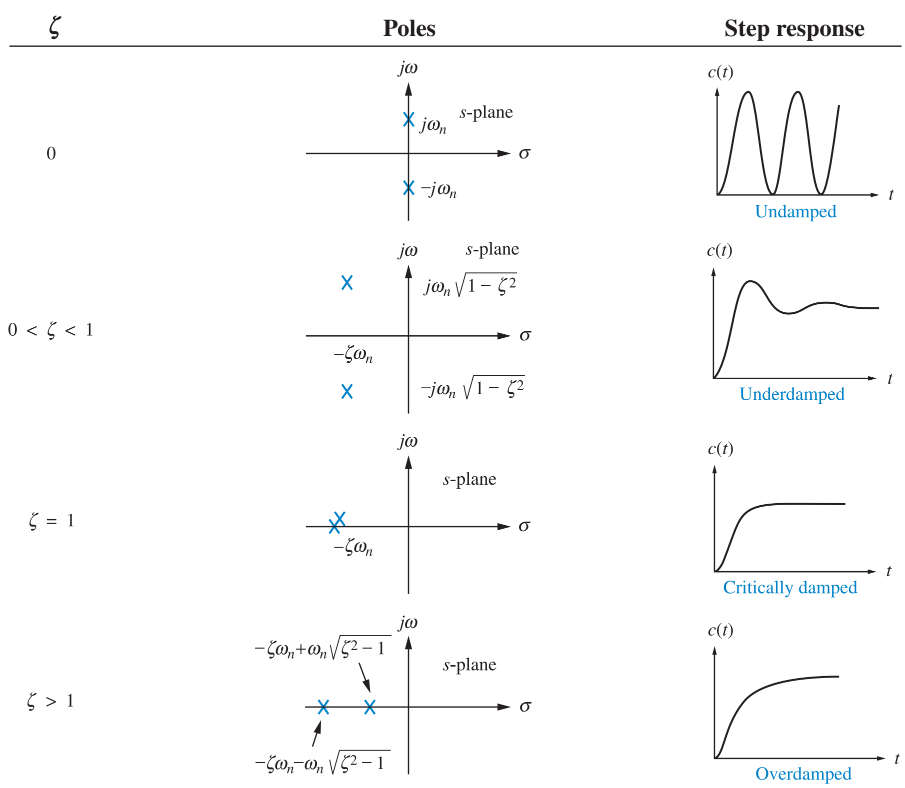


FIGURE 4.11 Second-order response as a function of damping ratio

[4] The student should verify Figure 4.11 as an exercise.

In the following example we find the numerical value of ζ and determine the nature of the transient response.

Example 4.4

Characterizing Response from the Value of ζ

PROBLEM: For each of the systems shown in Figure 4.12, find the value of ζ and report the kind of response expected.

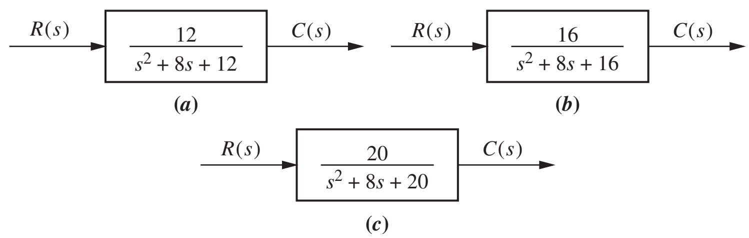


FIGURE 4.12 Systems for Example 4.4

SOLUTION: First match the form of these systems to the forms shown in Eqs. (4.16) and (4.22). Since $a = 2\zeta\omega_n$ and $\omega_n = \sqrt{b}$,

$$\zeta = \frac{a}{2\sqrt{b}} \tag{4.25}$$

Using the values of a and b from each of the systems of Figure 4.12, we find $\zeta = 1.155$ for system (a), which is thus overdamped, since $\zeta > 1$; $\zeta = 1$ for system (b), which is thus critically damped; and $\zeta = 0.894$ for system (c), which is thus underdamped, since $\zeta < 1$.

Skill-Assessment Exercise 4.4

PROBLEM: For each of the transfer functions in Skill-Assessment Exercise 4.3, do the following: (1) Find the values of ζ and ω_n; (2) characterize the nature of the response.

ANSWERS:

a. $\zeta = 0.3$, $\omega_n = 20$; system is underdamped

b. $\zeta = 1.5$, $\omega_n = 30$; system is overdamped

c. $\zeta = 1$, $\omega_n = 15$; system is critically damped

d. $\zeta = 0$, $\omega_n = 25$; system is undamped

The complete solution is located at www.wiley.com/college/nise.

This section defined two specifications, or parameters, of second-order systems: natural frequency, ω_n, and damping ratio, ζ. We saw that the nature of the response obtained was related to the value of ζ. Variations of damping ratio alone yield the complete range of overdamped, critically damped, underdamped, and undamped responses.

4.6 Underdamped Second-Order Systems

Now that we have generalized the second-order transfer function in terms of ζ and ω_n, let us analyze the step response of an *underdamped* second-order system. Not only will this response be found in terms of ζ and ω_n, but more specifications indigenous to the underdamped case will be defined. The underdamped second-order system, a common model for physical problems, displays unique behavior that must be itemized; a detailed description of the underdamped response is necessary for both analysis and design. Our first objective is to define transient specifications associated with underdamped responses. Next we relate these specifications to the pole location, drawing an association between pole location and the form of the underdamped second-order response. Finally, we tie the pole location to system parameters, thus closing the loop: Desired response generates required system components.

Let us begin by finding the step response for the general second-order system of Eq. (4.22). The transform of the response, $C(s)$, is the transform of the input times the transfer function, or

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.26}$$

where it is assumed that $\zeta < 1$ (the underdamped case). Expanding by partial fractions, using the methods described in Section 2.2, Case 3, yields

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \tag{4.27}$$

Taking the inverse Laplace transform, which is left as an exercise for the student, produces

$$\boxed{\begin{aligned} c(t) &= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right) \\ &= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi) \end{aligned}} \tag{4.28}$$

where $\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2})$.

A plot of this response appears in Figure 4.13 for various values of ζ, plotted along a time axis normalized to the natural frequency. We now see the relationship between the value of ζ and the type of response obtained: The lower the value of ζ, the more oscillatory the response. The natural frequency is a time-axis scale factor and does not affect the nature of the response other than to scale it in time.

We have defined two parameters associated with second-order systems, ζ and ω_n. Other parameters associated with the underdamped response are rise time, peak time, percent overshoot, and settling time. These specifications are defined as follows (see also Figure 4.14):

1. *Rise time*, T_r. The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. *Peak time*, T_P. The time required to reach the first, or maximum, peak.

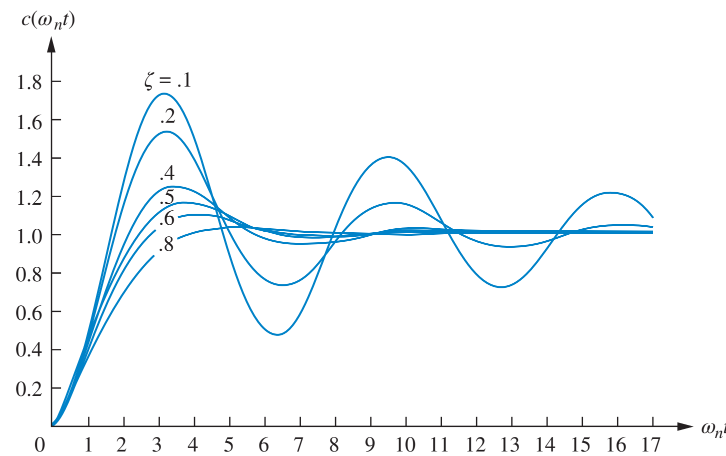


FIGURE 4.13 Second-order underdamped responses for damping ratio values

3. *Percent overshoot, %OS*. The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. *Settling time*, T_s. The time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady-state value.

Notice that the definitions for settling time and rise time are basically the same as the definitions for the first-order response. All definitions are also valid for systems of order higher than 2, although analytical expressions for these parameters cannot be found unless the response of the higher-order system can be approximated as a second-order system, which we do in Sections 4.7 and 4.8.

Rise time, peak time, and settling time yield information about the speed of the transient response. This information can help a designer determine if the speed and the nature of the response do or do not degrade the performance of the system. For example, the speed of an entire computer system depends on the time it takes for a hard drive head to reach steady state and read data; passenger comfort depends in part on the

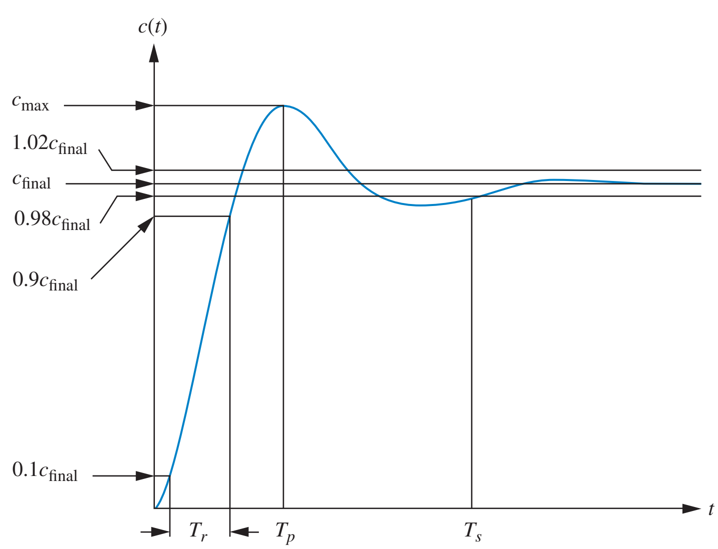


FIGURE 4.14 Second-order underdamped response specifications

suspension system of a car and the number of oscillations it goes through after hitting a bump.

We now evaluate T_p, $\%OS$, and T_s as functions of ζ and ω_n. Later in this chapter we relate these specifications to the location of the system poles. A precise analytical expression for rise time cannot be obtained; thus, we present a plot and a table showing the relationship between ζ and rise time.

Evaluation of T_p

T_p is found by differentiating $c(t)$ in Eq. (4.28) and finding the first zero crossing after $t = 0$. This task is simplified by "differentiating" in the frequency domain by using Item 7 of Table 2.2. Assuming zero initial conditions and using Eq. (4.26), we get

$$\mathscr{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.29}$$

Completing squares in the denominator, we have

$$\mathscr{L}[\dot{c}(t)] = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{\dfrac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \tag{4.30}$$

Therefore,

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n\sqrt{1-\zeta^2}\,t \tag{4.31}$$

Setting the derivative equal to zero yields

$$\omega_n\sqrt{1-\zeta^2}\,t = n\pi \tag{4.32}$$

or

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{4.33}$$

Each value of n yields the time for local maxima or minima. Letting $n = 0$ yields $t = 0$, the first point on the curve in Figure 4.14 that has zero slope. The first peak, which occurs at the peak time, T_p, is found by letting $n = 1$ in Eq. (4.33):

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \tag{4.34}$$

Evaluation of $\%OS$

From Figure 4.14 the percent overshoot, $\%OS$, is given by

$$\%OS = \frac{c_{\max} - c_{\text{final}}}{c_{\text{final}}} \times 100 \tag{4.35}$$

The term $c_{\max}$ is found by evaluating $c(t)$ at the peak time, $c(T_p)$. Using Eq. (4.34) for T_p and substituting into Eq. (4.28) yields

$$\begin{aligned} c_{\max} = c(T_p) &= 1 - e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) \\ &= 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \end{aligned} \tag{4.36}$$

For the unit step used for Eq. (4.28),

$$c_{\text{final}} = 1 \tag{4.37}$$

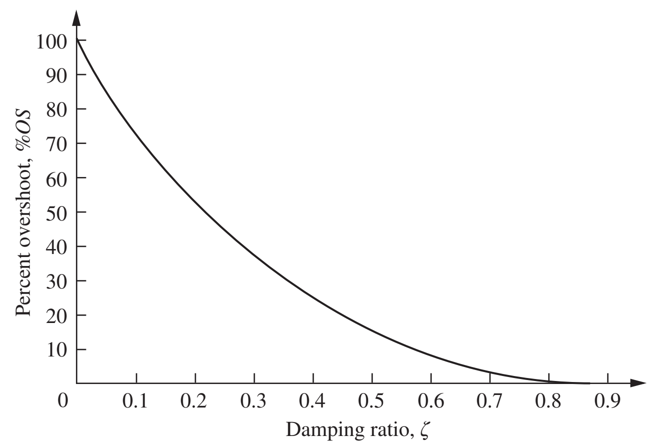


FIGURE 4.15 Percent overshoot versus damping ratio

Substituting Eqs. (4.36) and (4.37) into Eq. (4.35), we finally obtain

$$\boxed{\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100} \tag{4.38}$$

Notice that the percent overshoot is a function only of the damping ratio, ζ.

Whereas Eq. (4.38) allows one to find $\%OS$ given ζ, the inverse of the equation allows one to solve for ζ given $\%OS$. The inverse is given by

$$\boxed{\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}} \tag{4.39}$$

The derivation of Eq. (4.39) is left as an exercise for the student. Equation (4.38) (or, equivalently, (4.39)) is plotted in Figure 4.15.

Evaluation of T_s

In order to find the settling time, we must find the time for which $c(t)$ in Eq. (4.28) reaches and stays within $\pm 2\%$ of the steady-state value, c_{final}. Using our definition, the settling time is the time it takes for the amplitude of the decaying sinusoid in Eq. (4.28) to reach 0.02, or

$$e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}} = 0.02 \tag{4.40}$$

This equation is a conservative estimate, since we are assuming that $\cos(\omega_n\sqrt{1-\zeta^2}t - \phi) = 1$ at the settling time. Solving Eq. (4.40) for t, the settling time is

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n} \tag{4.41}$$

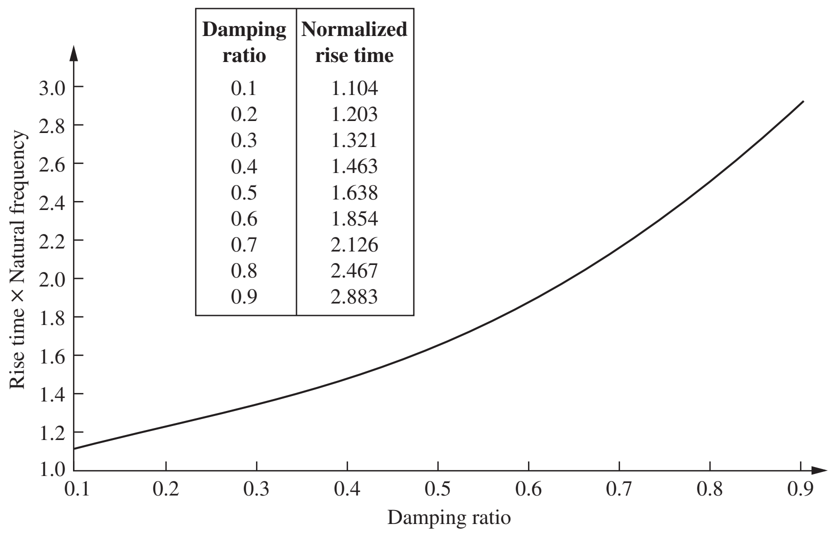
You can verify that the numerator of Eq. (4.41) varies from 3.91 to 4.74 as ζ varies from 0 to 0.9. Let us agree on an approximation for the settling time that will be used for all values of ζ; let it be

$$\boxed{T_s = \frac{4}{\zeta\omega_n}} \tag{4.42}$$

Evaluation of T_r

A precise analytical relationship between rise time and damping ratio, ζ, cannot be found. However, using a computer and Eq. (4.28), the rise time can be found. We first designate $\omega_n t$

as the normalized time variable and select a value for ζ. Using the computer, we solve for the values of $\omega_n t$ that yield $c(t) = 0.9$ and $c(t) = 0.1$. Subtracting the two values of $\omega_n t$ yields the normalized rise time, $\omega_n T_r$, for that value of ζ. Continuing in like fashion with other values of ζ, we obtain the results plotted in Figure 4.16.[5] Let us look at an example.



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

FIGURE 4.16 Normalized rise time versus damping ratio for a second-order underdamped response

Example 4.5

Finding T_p, $\%OS$, T_s, and T_r from a Transfer Function

PROBLEM: Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100} \tag{4.43}$$

find T_p, $\%OS$, T_s, and T_r.

SOLUTION: ω_n and ζ are calculated as 10 and 0.75, respectively. Now substitute ζ and ω_n into Eqs. (4.34), (4.38), and (4.42) and find, respectively, that $T_p = 0.475$ second, $\%OS = 2.838$, and $T_s = 0.533$ second. Using the table in Figure 4.16, the normalized rise time is approximately 2.3 seconds. Dividing by ω_n yields $T_r = 0.23$ second. This problem demonstrates that we can find T_p, $\%OS$, T_s, and T_r without the tedious task of taking an inverse Laplace transform, plotting the output response, and taking measurements from the plot.

Virtual Experiment 4.2
Second-Order System Response

Put theory into practice studying the effect that natural frequency and damping ratio have on controlling the speed response of the Quanser Linear Servo in LabVIEW. This concept is applicable to automobile cruise controls or speed controls of subways or trucks.

Virtual experiments are found on Learning Space.

[5] Figure 4.16 can be approximated by the following polynomials: $\omega_n T_r = 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1$ (maximum error less than $\frac{1}{2}\%$ for $0 < \zeta < 0.9$), and $\zeta = 0.115(\omega_n T_r)^3 - 0.883(\omega_n T_r)^2 + 2.504(\omega_n T_r) - 1.738$ (maximum error less than 5% for $0.1 < \zeta < 0.9$). The polynomials were obtained using MATLAB's **polyfit** function.

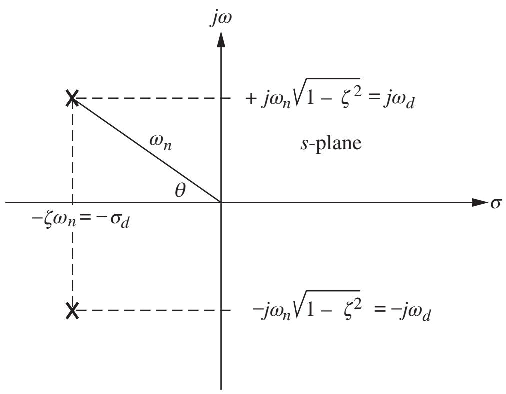


FIGURE 4.17 Pole plot for an underdamped second-order system

We now have expressions that relate peak time, percent overshoot, and settling time to the natural frequency and the damping ratio. Now let us relate these quantities to the location of the poles that generate these characteristics.

The pole plot for a general, underdamped second-order system, previously shown in Figure 4.11, is reproduced and expanded in Figure 4.17 for focus. We see from the Pythagorean theorem that the radial distance from the origin to the pole is the natural frequency, ω_n, and the $\cos\theta = \zeta$.

Now, comparing Eqs. (4.34) and (4.42) with the pole location, we evaluate peak time and settling time in terms of the pole location. Thus,

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \tag{4.44}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{\pi}{\sigma_d} \tag{4.45}$$

where ω_d is the imaginary part of the pole and is called the *damped frequency of oscillation*, and σ_d is the magnitude of the real part of the pole and is the *exponential damping frequency*.

Equation (4.44) shows that T_p is inversely proportional to the imaginary part of the pole. Since horizontal lines on the s-plane are lines of constant imaginary value, they are also lines of constant peak time. Similarly, Eq. (4.45) tells us that settling time is inversely proportional to the real part of the pole. Since vertical lines on the s-plane are lines of constant real value, they are also lines of constant settling time. Finally, since $\zeta = \cos\theta$, radial lines are lines of constant ζ. Since percent overshoot is only a function of ζ, radial lines are thus lines of constant percent overshoot, $\%OS$. These concepts are depicted in Figure 4.18, where lines of constant T_p, T_s, and $\%OS$ are labeled on the s-plane.

At this point, we can understand the significance of Figure 4.18 by examining the actual step response of comparative systems. Depicted in Figure 4.19(a) are the step responses as the poles are moved in a vertical direction, keeping the real part the same.

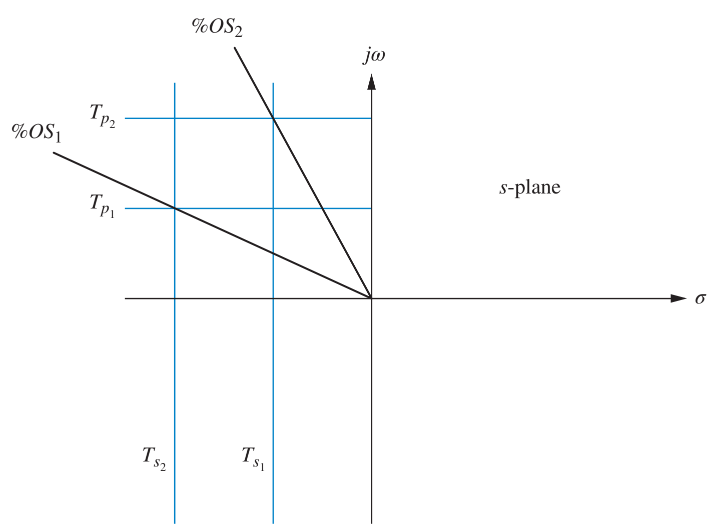


FIGURE 4.18 Lines of constant peak time, T_p, settling time, T_s, and percent overshoot, $\%OS$. Note: $T_{s_2} < T_{s_1}$; $T_{p2} < T_{p1}$; $\%OS_1 < \%OS_2$.

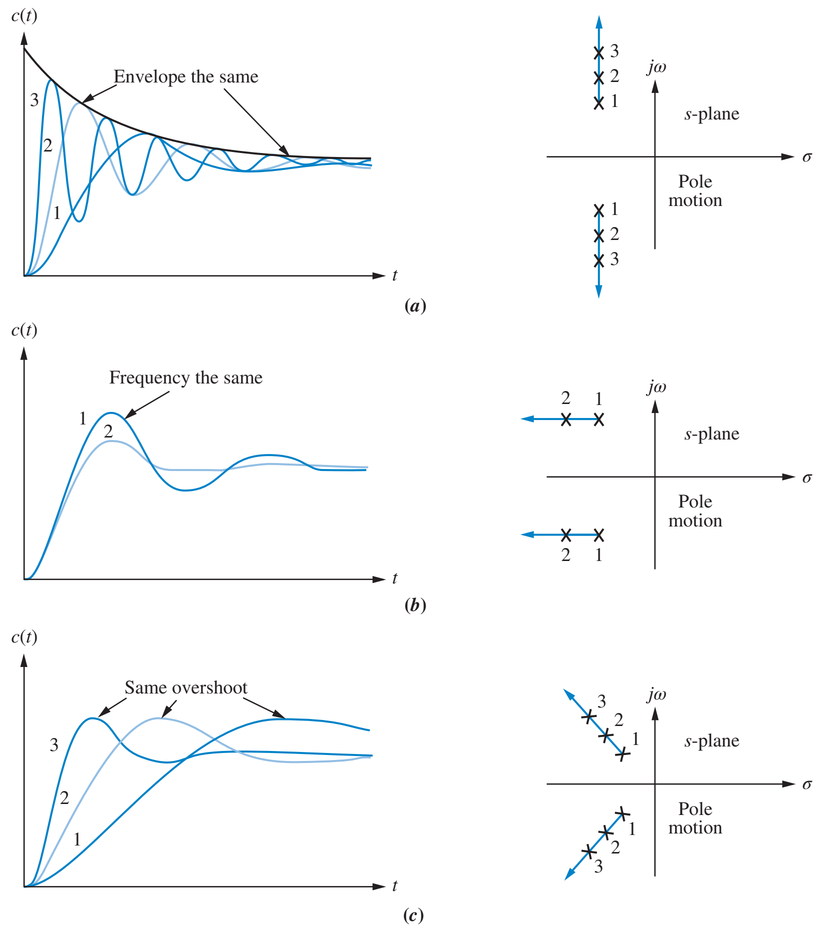


FIGURE 4.19 Step responses of second-order underdamped systems as poles move: **a.** with constant real part; **b.** with constant imaginary part; **c.** with constant damping ratio

As the poles move in a vertical direction, the frequency increases, but the envelope remains the same since the real part of the pole is not changing. The figure shows a constant exponential envelope, even though the sinusoidal response is changing frequency. Since all curves fit under the same exponential decay curve, the settling time is virtually the same for all waveforms. Note that as overshoot increases, the rise time decreases.

Let us move the poles to the right or left. Since the imaginary part is now constant, movement of the poles yields the responses of Figure 4.19(*b*). Here the frequency is constant over the range of variation of the real part. As the poles move to the left, the response damps out more rapidly, while the frequency remains the same. Notice that the peak time is the same for all waveforms because the imaginary part remains the same.

Moving the poles along a constant radial line yields the responses shown in Figure 4.19(*c*). Here the percent overshoot remains the same. Notice also that the responses look exactly alike, except for their speed. The farther the poles are from the origin, the more rapid the response.

We conclude this section with some examples that demonstrate the relationship between the pole location and the specifications of the second-order underdamped response. The first example covers analysis. The second example is a simple design problem consisting of a physical system whose component values we want to design to meet a transient response specification. An animation PowerPoint presentation (PPT) demonstrating second-order principles is available for instructors at www.wiley.com/college/nise. See *Second-Order Step Response*.

Example 4.6

Finding T_p, $\%OS$, and T_s from Pole Location

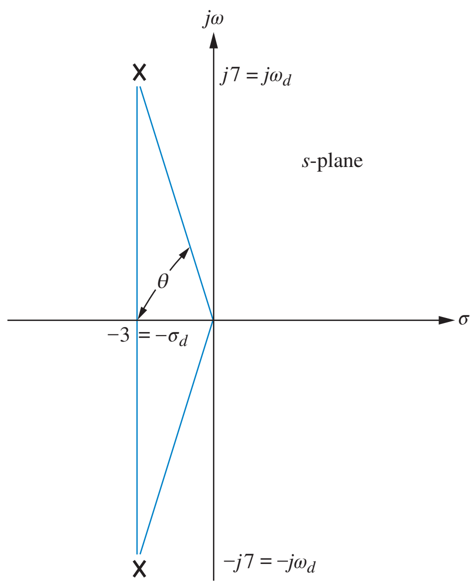


FIGURE 4.20 Pole plot for Example 4.6

PROBLEM: Given the pole plot shown in Figure 4.20, find ζ, ω_n, T_p, $\%OS$, and T_s.

SOLUTION: The damping ratio is given by $\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$. The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$. The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second} \tag{4.46}$$

The percent overshoot is

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26\% \tag{4.47}$$

The approximate settling time is

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds} \tag{4.48}$$

MATLAB
ML

Students who are using MATLAB should now run ch4p1 in Appendix B. You will learn how to generate a second-order polynomial from two complex poles as well as extract and use the coefficients of the polynomial to calculate T_p, $\%OS$, and T_s. This exercise uses MATLAB to solve the problem in Example 4.6.

Example 4.7

Transient Response Through Component Design

Design
D

PROBLEM: Given the system shown in Figure 4.21, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque $T(t)$.

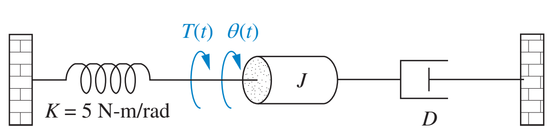


FIGURE 4.21 Rotational mechanical system for Example 4.7

SOLUTION: First, the transfer function for the system is

$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}} \tag{4.49}$$

From the transfer function,

$$\omega_n = \sqrt{\frac{K}{J}} \tag{4.50}$$

and

$$2\zeta\omega_n = \frac{D}{J} \tag{4.51}$$

But, from the problem statement,

$$T_s = 2 = \frac{4}{\zeta\omega_n} \tag{4.52}$$

or $\zeta\omega_n = 2$. Hence,

$$2\zeta\omega_n = 4 = \frac{D}{J} \tag{4.53}$$

Also, from Eqs. (4.50) and (4.52),

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}} \tag{4.54}$$

From Eq. (4.39), a 20% overshoot implies $\zeta = 0.456$. Therefore, from Eq. (4.54),

$$\zeta = 2\sqrt{\frac{J}{K}} = 0.456 \tag{4.55}$$

Hence,

$$\frac{J}{K} = 0.052 \tag{4.56}$$

From the problem statement, $K = 5$ N-m/rad. Combining this value with Eqs. (4.53) and (4.56), $D = 1.04$ N-m-s/rad, and $J = 0.26\ \text{kg-m}^2$.

Second-Order Transfer Functions via Testing

Just as we obtained the transfer function of a first-order system experimentally, we can do the same for a system that exhibits a typical underdamped second-order response. Again, we can measure the laboratory response curve for percent overshoot and settling time, from which we can find the poles and hence the denominator. The numerator can be found, as in the first-order system, from a knowledge of the measured and expected steady-state values. A problem at the end of the chapter illustrates the estimation of a second-order transfer function from the step response.

Skill-Assessment Exercise 4.5

TryIt 4.1

Use the following MATLAB statements to calculate the answers to Skill-Assessment Exercise 4.5. Ellipses mean code continues on next line.

```
numg=361;
deng=[1 16 361];
omegan=sqrt(deng(3)...
  /deng(1))
zeta=(deng(2)/deng(1))...
  /(2*omegan)
Ts=4/(zeta*omegan)
Tp=pi/(omegan*sqrt...
  (1-zeta^2))
pos=100*exp(-zeta*...
  pi/sqrt(1-zeta^2))
Tr=(1.768*zeta^3 -...
  0.417*zeta^2+1.039*...
  zeta+1)/omegan
```

PROBLEM: Find ζ, ω_n, T_s, T_p, T_r, and $\%OS$ for a system whose transfer function is $G(s) = \dfrac{361}{s^2 + 16s + 361}$.

ANSWERS:

$\zeta = 0.421$, $\omega_n = 19$, $T_s = 0.5$ s, $T_p = 0.182$ s, $T_r = 0.079$ s, and $\%OS = 23.3\%$.

The complete solution is located at www.wiley.com/college/nise.

Now that we have analyzed systems with two poles, how does the addition of another pole affect the response? We answer this question in the next section.

4.7 System Response with Additional Poles

In the last section, we analyzed systems with one or two poles. It must be emphasized that the formulas describing percent overshoot, settling time, and peak time were derived only for a system with two complex poles and no zeros. If a system such as that shown in Figure 4.22 has more than two poles or has zeros, we cannot use the formulas to calculate the performance specifications that we derived. However, under certain conditions, a system with more than two poles or with zeros can be approximated as a second-order system that has just two complex *dominant poles*. Once we justify this approximation, the formulas for percent overshoot, settling time, and peak time can be applied to these higher-order systems by using the location of the dominant poles. In this section, we investigate the effect of an additional pole on the second-order response. In the next section, we analyze the effect of adding a zero to a two-pole system.

Let us now look at the conditions that would have to exist in order to approximate the behavior of a three-pole system as that of a two-pole system. Consider a three-pole system with complex poles and a third pole on the real axis. Assuming that the complex poles are at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ and the real pole is at $-\alpha_r$, the step response of the system can be determined from a partial-fraction expansion. Thus, the output transform is

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r} \tag{4.57}$$

or, in the time domain,

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t} \tag{4.58}$$

The component parts of $c(t)$ are shown in Figure 4.23 for three cases of α_r. For Case I, $\alpha_r = \alpha_{r_1}$ and is not much larger than $\zeta\omega_n$; for Case II, $\alpha_r = \alpha_{r_2}$ and is much larger than $\zeta\omega_n$; and for Case III, $\alpha_r = \infty$.

FIGURE 4.22 Robot follows input commands from a human trainer

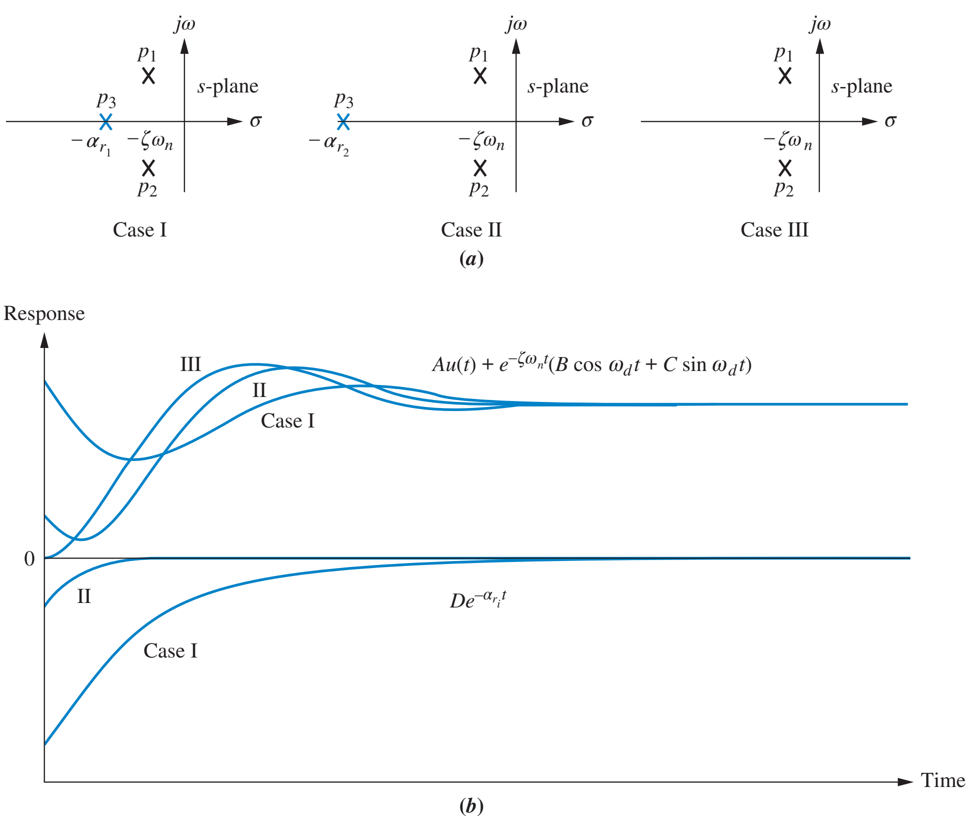


FIGURE 4.23 Component responses of a three-pole system: **a.** pole plot; **b.** component responses: Nondominant pole is near dominant second-order pair (Case I), far from the pair (Case II), and at infinity (Case III)

Let us direct our attention to Eq. (4.58) and Figure 4.23. If $\alpha_r \gg \zeta\omega_n$ (Case II), the pure exponential will die out much more rapidly than the second-order underdamped step response. If the pure exponential term decays to an insignificant value at the time of the first overshoot, such parameters as percent overshoot, settling time, and peak time will be generated by the second-order underdamped step response component. Thus, the total response will approach that of a pure second-order system (Case III).

If α_r is not much greater than $\zeta\omega_n$ (Case I), the real pole's transient response will not decay to insignificance at the peak time or settling time generated by the second-order pair. In this case, the exponential decay is significant, and the system cannot be represented as a second-order system.

The next question is, How much farther from the dominant poles does the third pole have to be for its effect on the second-order response to be negligible? The answer of course depends on the accuracy for which you are looking. However, this book assumes that the exponential decay is negligible after five time constants. Thus, if the real pole is five times farther to the left than the dominant poles, we assume that the system is represented by its dominant second-order pair of poles.

What about the magnitude of the exponential decay? Can it be so large that its contribution at the peak time is not negligible? We can show, through a partial-fraction expansion, that the residue of the third pole, in a three-pole system with dominant second-order poles and no zeros, will actually decrease in magnitude as the third pole is moved farther into the left half-plane. Assume a step response, $C(s)$, of a three-pole system:

$$C(s) = \frac{bc}{s(s^2+as+b)(s+c)} = \frac{A}{s} + \frac{Bs+C}{s^2+as+b} + \frac{D}{s+c} \tag{4.59}$$

where we assume that the nondominant pole is located at $-c$ on the real axis and that the steady-state response approaches unity. Evaluating the constants in the numerator of each term,

$$A = 1; \qquad B = \frac{ca - c^2}{c^2 + b - ca} \tag{4.60a}$$

$$C = \frac{ca^2 - c^2 a - bc}{c^2 + b - ca}; \quad D = \frac{-b}{c^2 + b - ca} \tag{4.60b}$$

As the nondominant pole approaches ∞, or $c \rightarrow \infty$,

$$A = 1;\ B = -1;\ C = -a;\ D = 0 \tag{4.61}$$

Thus, for this example, D, the residue of the nondominant pole and its response, becomes zero as the nondominant pole approaches infinity.

The designer can also choose to forgo extensive residue analysis, since all system designs should be simulated to determine final acceptance. In this case, the control systems engineer can use the "five times" rule of thumb as a necessary but not sufficient condition to increase the confidence in the second-order approximation during design, but then simulate the completed design.

Let us look at an example that compares the responses of two different three-pole systems with that of a second-order system.

Example 4.8

Comparing Responses of Three-Pole Systems

PROBLEM: Find the step response of each of the transfer functions shown in Eqs. (4.62) through (4.64) and compare them.

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542} \tag{4.62}$$

$$T_2(s) = \frac{245.42}{(s+10)(s^2 + 4s + 24.542)} \tag{4.63}$$

$$T_3(s) = \frac{73.626}{(s+3)(s^2 + 4s + 24.542)} \tag{4.64}$$

SOLUTION: The step response, $C_i(s)$, for the transfer function, $T_i(s)$, can be found by multiplying the transfer function by $1/s$, a step input, and using partial-fraction expansion followed by the inverse Laplace transform to find the response, $c_i(t)$. With the details left as an exercise for the student, the results are

$$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8°) \tag{4.65}$$

$$c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34°) \tag{4.66}$$

$$c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63°) \tag{4.67}$$

The three responses are plotted in Figure 4.24. Notice that $c_2(t)$, with its third pole at -10 and farthest from the dominant poles, is the better approximation of $c_1(t)$, the pure second-order system response; $c_3(t)$, with a third pole close to the dominant poles, yields the most error.

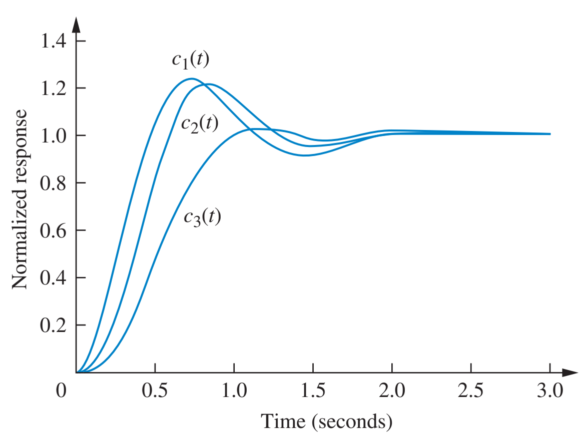


FIGURE 4.24 Step responses of system $T_1(s)$, system $T_2(s)$, and system $T_3(s)$

MATLAB

ML

Students who are using MATLAB should now run ch4p2 in Appendix B. You will learn how to generate a step response for a transfer function and how to plot the response directly or collect the points for future use. The example shows how to collect the points and then use them to create a multiple plot, title the graph, and label the axes and curves to produce the graph in Figure 4.24 to solve Example 4.8.

Simulink

SL

System responses can alternately be obtained using Simulink. Simulink is a software package that is integrated with MATLAB to provide a graphical user interface (GUI) for defining systems and generating responses. The reader is encouraged to study Appendix C, which contains a tutorial on Simulink as well as some examples. One of the illustrative examples, Example C.1, solves Example 4.8 using Simulink.

GUI Tool

GUIT

Another method to obtain systems responses is through the use of MATLAB's LTI Viewer. An advantage of the LTI Viewer is that it displays the values of settling time, peak time, rise time, maximum response, and the final value on the step response plot. The reader is encouraged to study Appendix E at www.wiley.com/college/nise, which contains a tutorial on the LTI Viewer as well as some examples. Example E.1 solves Example 4.8 using the LTI Viewer.

Skill-Assessment Exercise 4.6

TryIt 4.2

Use the following MATLAB and Control System Toolbox statements to investigate the effect of the additional pole in Skill-Assessment Exercise 4.6(*a*). Move the higher-order pole originally at −15 to other values by changing "a" in the code.

```
a=15
numga=100*a;
denga=conv([1 a],...
 [1 4 100]);
Ta=tf(numga,denga);
numg=100;
deng=[1 4 100];
T=tf(numg,deng);
step(Ta,'.',T,'-')
```

PROBLEM: Determine the validity of a second-order approximation for each of these two transfer functions:

a. $G(s) = \dfrac{700}{(s+15)(s^2+4s+100)}$

b. $G(s) = \dfrac{360}{(s+4)(s^2+2s+90)}$

ANSWERS:

a. The second-order approximation is valid.

b. The second-order approximation is not valid.

The complete solution is located at www.wiley.com/college/nise.

4.8 System Response with Zeros

Now that we have seen the effect of an additional pole, let us add a zero to the second-order system. In Section 4.2, we saw that the zeros of a response affect the residue, or amplitude, of a response component but do not affect the nature of the response—exponential, damped sinusoid, and so on. In this section, we add a real-axis zero to a two-pole system. The zero will be added first in the left half-plane and then in the right half-plane and its effects noted and analyzed. We conclude the section by talking about pole-zero cancellation.

Starting with a two-pole system with poles at $(-1 \pm j2.828)$, we consecutively add zeros at -3, -5, and -10. The results, normalized to the steady-state value, are plotted in Figure 4.25. We can see that the closer the zero is to the dominant poles, the greater its effect on the transient response. As the zero moves away from the dominant poles, the response approaches that of the two-pole system. This analysis can be reasoned via the partial-fraction expansion. If we assume a group of poles and a zero far from the poles, the residue of each pole will be affected the same by the zero. Hence, the relative amplitudes remain appreciably the same. For example, assume the partial-fraction expansion shown in Eq. (4.68):

$$\begin{aligned} T(s) = \frac{(s+a)}{(s+b)(s+c)} &= \frac{A}{s+b} + \frac{B}{s+c} \\ &= \frac{(-b+a)/(-b+c)}{s+b} + \frac{(-c+a)/(-c+b)}{s+c} \end{aligned} \tag{4.68}$$

If the zero is far from the poles, then a is large compared to b and c, and

$$T(s) \approx a\left[\frac{1/(-b+c)}{s+b} + \frac{1/(-c+b)}{s+c}\right] = \frac{a}{(s+b)(s+c)} \tag{4.69}$$

Hence, the zero looks like a simple gain factor and does not change the relative amplitudes of the components of the response.

Another way to look at the effect of a zero, which is more general, is as follows (*Franklin, 1991*): Let $C(s)$ be the response of a system, $T(s)$, with unity in the numerator. If we add a zero to the transfer function, yielding $(s+a)T(s)$, the Laplace transform of the response will be

$$\boxed{(s+a)C(s) = sC(s) + aC(s)} \tag{4.70}$$

Thus, the response of a system with a zero consists of two parts: the derivative of the original response and a scaled version of the original response. If a, the negative of the zero, is very large, the Laplace transform of the response is approximately $aC(s)$, or a scaled version of the original response. If a is not very large, the response has an additional component consisting of the derivative of the original response. As a becomes smaller, the derivative term contributes more to the response and has a greater effect. For step responses, the derivative is typically positive at the start of a step response. Thus, for small values of a, we can expect more overshoot in second-order systems because the derivative term will be additive around the first overshoot. This reasoning is borne out by Figure 4.25.

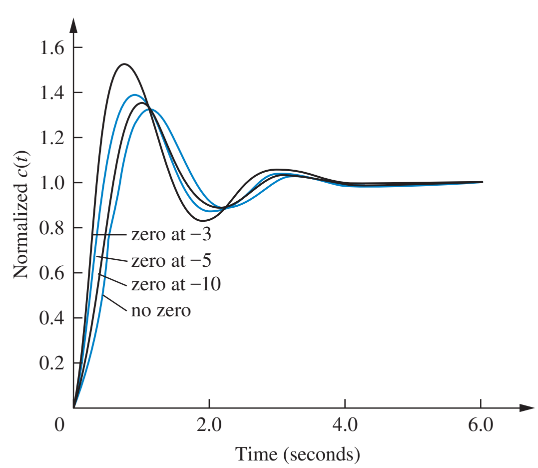


FIGURE 4.25 Effect of adding a zero to a two-pole system

TryIt 4.3

Use the following MATLAB and Control System Toolbox statements to generate Figure 4.25.

```
deng=[1 2 9];
Ta=tf([1 3]*9/3,deng);
Tb=tf([1 5]*9/5,deng);
Tc=tf([1 10]*9/10,deng);
T=tf(9,deng);
step(T,Ta,Tb,Tc)
text(0.5,0.6,'no zero')
text(0.4,0.7,...
 'zero at -10')
text(0.35,0.8,...
 'zero at -5')
text(0.3,0.9,'zero at -3')
```

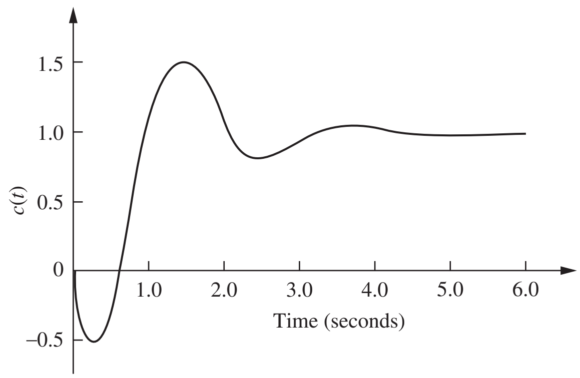


FIGURE 4.26 Step response of a nonminimum-phase system

An interesting phenomenon occurs if a is negative, placing the zero in the right half-plane. From Eq. (4.70) we see that the derivative term, which is typically positive initially, will be of opposite sign from the scaled response term. Thus, if the derivative term, $sC(s)$, is larger than the scaled response, $aC(s)$, the response will initially follow the derivative in the opposite direction from the scaled response. The result for a second-order system is shown in Figure 4.26, where the sign of the input was reversed to yield a positive steady-state value. Notice that the response begins to turn toward the negative direction even though the final value is positive. A system that exhibits this phenomenon is known as a *nonminimum-phase* system. If a motorcycle or airplane was a nonminimum-phase system, it would initially veer left when commanded to steer right.

Let us now look at an example of an electrical nonminimum-phase network.

Example 4.9

Transfer Function of a Nonminimum-Phase System

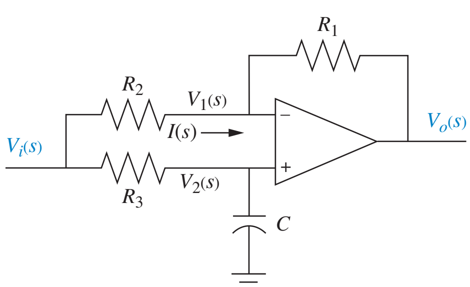


FIGURE 4.27 Nonminimum-phase electric circuit[6]

PROBLEM:

a. Find the transfer function, $V_o(s)/V_i(s)$ for the operational amplifier circuit shown in Figure 4.27.

b. If $R_1 = R_2$, this circuit is known as an all-pass filter, since it passes sine waves of a wide range of frequencies without attenuating or amplifying their magnitude (*Dorf, 1993*). We will learn more about frequency response in Chapter 10. For now, let $R_1 = R_2$, $R_3C = 1/10$, and find the step response of the filter. Show that component parts of the response can be identified with those in Eq. (4.70).

SOLUTION:

a. Remembering from Chapter 2 that the operational amplifier has a high input impedance, the current, $I(s)$, through R_1 and R_2, is the same and is equal to

$$I(s) = \frac{V_i(s) - V_o(s)}{R_1 + R_2} \tag{4.71}$$

[6] Adapted from Dorf, R. C. *Introduction to Electric Circuits, 2nd ed.* (New York: John Wiley & Sons, 1989, 1993), p. 583.

Also,

$$V_o(s) = A(V_2(s) - V_1(s)) \tag{4.72}$$

But,

$$V_1(s) = I(s)R_1 + V_o(s) \tag{4.73}$$

Substituting Eq. (4.71) into (4.73),

$$V_1(s) = \frac{1}{R_1 + R_2}(R_1 V_i(s) + R_2 V_0(s)) \tag{4.74}$$

Using voltage division,

$$V_2(s) = V_i(s)\frac{1/Cs}{R_3 + \dfrac{1}{Cs}} \tag{4.75}$$

Substituting Eqs. (4.74) and (4.75) into Eq. (4.72) and simplifying yields

$$\frac{V_o(s)}{V_i(s)} = \frac{A(R_2 - R_1R_3Cs)}{(R_3Cs + 1)(R_1 + R_2(1 + A))} \tag{4.76}$$

Since the operational amplifier has a large gain, A, let A approach infinity. Thus, after simplification

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 - R_1R_3Cs}{R_2R_3Cs + R_2} = -\frac{R_1}{R_2}\frac{\left(s - \dfrac{R_2}{R_1R_3C}\right)}{\left(s + \dfrac{1}{R_3C}\right)} \tag{4.77}$$

b. Letting $R_1 = R_2$ and $R_3C = 1/10$,

$$\frac{V_o(s)}{V_i(s)} = \frac{\left(s - \dfrac{1}{R_3C}\right)}{\left(s + \dfrac{1}{R_3C}\right)} = -\frac{(s - 10)}{(s + 10)} \tag{4.78}$$

For a step input, we evaluate the response as suggested by Eq. (4.70):

$$C(s) = -\frac{(s - 10)}{s(s + 10)} = -\frac{1}{s + 10} + 10\,\frac{1}{s(s + 10)} = sC_o(s) - 10C_o(s) \tag{4.79}$$

where

$$C_o(s) = -\frac{1}{s(s + 10)} \tag{4.80}$$

is the Laplace transform of the response without a zero. Expanding Eq. (4.79) into partial fractions,

$$C(s) = -\frac{1}{s + 10} + 10\frac{1}{s(s + 10)} = -\frac{1}{s + 10} + \frac{1}{s} - \frac{1}{s + 10} = \frac{1}{s} - \frac{2}{s + 10} \tag{4.81}$$

or the response with a zero is

$$c(t) = -e^{-10t} + 1 - e^{-10t} = 1 - 2e^{-10t} \tag{4.82}$$

Also, from Eq. (4.80),

$$C_o(s) = -\frac{1/10}{s} + \frac{1/10}{s+10} \tag{4.83}$$

or the response without a zero is

$$c_o(t) = -\frac{1}{10} + \frac{1}{10}e^{-10t} \tag{4.84}$$

The normalized responses are plotted in Figure 4.28. Notice the immediate reversal of the nonminimum-phase response, $c(t)$.

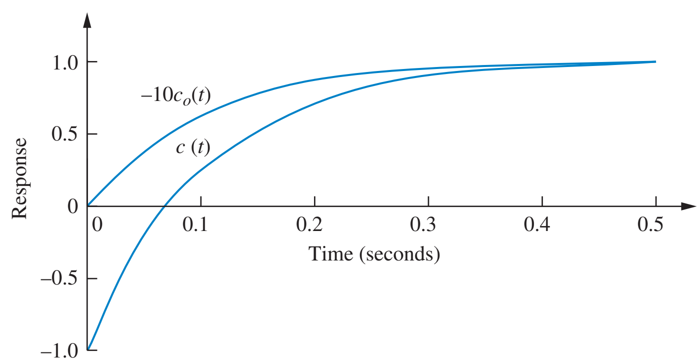


FIGURE 4.28 Step response of the nonminimum-phase network of Figure 4.27 ($c(t)$) and normalized step response of an equivalent network without the zero ($-10\,c_o(t)$)

We conclude this section by talking about pole-zero cancellation and its effect on our ability to make second-order approximations to a system. Assume a three-pole system with a zero as shown in Eq. (4.85). If the pole term, $(s + p_3)$, and the zero term, $(s + z)$, cancel out, we are left with

$$T(s) = \frac{K\cancel{(s+z)}}{\cancel{(s+p_3)}(s^2 + as + b)} \tag{4.85}$$

as a second-order transfer function. From another perspective, if the zero at $-z$ is very close to the pole at $-p_3$, then a partial-fraction expansion of Eq. (4.85) will show that the residue of the exponential decay is much smaller than the amplitude of the second-order response. Let us look at an example.

Example 4.10

Evaluating Pole-Zero Cancellation Using Residues

PROBLEM: For each of the response functions in Eqs. (4.86) and (4.87), determine whether there is cancellation between the zero and the pole closest to the zero. For any function for which pole-zero cancellation is valid, find the approximate response.

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)} \tag{4.86}$$

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)} \tag{4.87}$$

SOLUTION: The partial-fraction expansion of Eq. (4.86) is

$$C_1(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5} \tag{4.88}$$

The residue of the pole at −3.5, which is closest to the zero at −4, is equal to 1 and is not negligible compared to the other residues. Thus, a second-order step response approximation cannot be made for $C_1(s)$. The partial-fraction expansion for $C_2(s)$ is

$$C_2(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{0.033}{s+4.01} \tag{4.89}$$

The residue of the pole at −4.01, which is closest to the zero at −4, is equal to 0.033, about two orders of magnitude below any of the other residues. Hence, we make a second-order approximation by neglecting the response generated by the pole at −4.01:

$$C_2(s) \approx \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} \tag{4.90}$$

and the response $c_2(t)$ is approximately

$$c_2(t) \approx 0.87 - 5.3e^{-5t} + 4.4e^{-6t} \tag{4.91}$$

TryIt 4.4

Use the following MATLAB and Symbolic Math Toolbox statements to evaluate the effect of higher-order poles by finding the component parts of the time response of c_1(t) and c_2(t) in Example 4.10.

```
syms s
C1=26.25*(s+4)/...
  (s*(s+3.5)*...
  (s+5)*(s+6));
C2=26.25*(s+4)/...
  (s*(s+4.01)*...
  (s+5)*(s+6));
c1=ilaplace(C1);
'c1'
c1=vpa(c1,3)
c2=ilaplace(C2);
'c2'
c2=vpa(c2,3)
```

Skill-Assessment Exercise 4.7

PROBLEM: Determine the validity of a second-order step-response approximation for each transfer function shown below.

a. $G(s) = \dfrac{185.71(s+7)}{(s+6.5)(s+10)(s+20)}$

b. $G(s) = \dfrac{197.14(s+7)}{(s+6.9)(s+10)(s+20)}$

ANSWERS:

a. A second-order approximation is not valid.

b. A second-order approximation is valid.

The complete solution is located at www.wiley.com/college/nise.

In this section, we have examined the effects of additional transfer function poles and zeros upon the response. In the next section we add nonlinearities of the type discussed in Section 2.10 and see what effects they have on system response.

4.9 Effects of Nonlinearities upon Time Response

In this section, we qualitatively examine the effects of nonlinearities upon the time response of physical systems. In the following examples, we insert nonlinearities, such as saturation, dead zone, and backlash, as shown in Figure 2.46, into a system to show the effects of these nonlinearities upon the linear responses.

The responses were obtained using Simulink, a simulation software package that is integrated with MATLAB to provide a graphical user interface (GUI). Readers who would like to learn how to use Simulink to generate nonlinear responses should consult the Simulink tutorial in Appendix C. Simulink block diagrams are included with all responses that follow.

Let us assume the motor and load from the Antenna Control Case Study of Chapter 2 and look at the load angular velocity, $\omega_o(s)$, where $\omega_o(s) = 0.1\, s\theta_m(s) = 0.2083\, E_a(s)/(s + 1.71)$ from Eq. (2.208). If we drive the motor with a step input through an amplifier of unity gain that saturates at ± 5 volts, Figure 4.29 shows that the effect of amplifier saturation is to limit the obtained velocity.

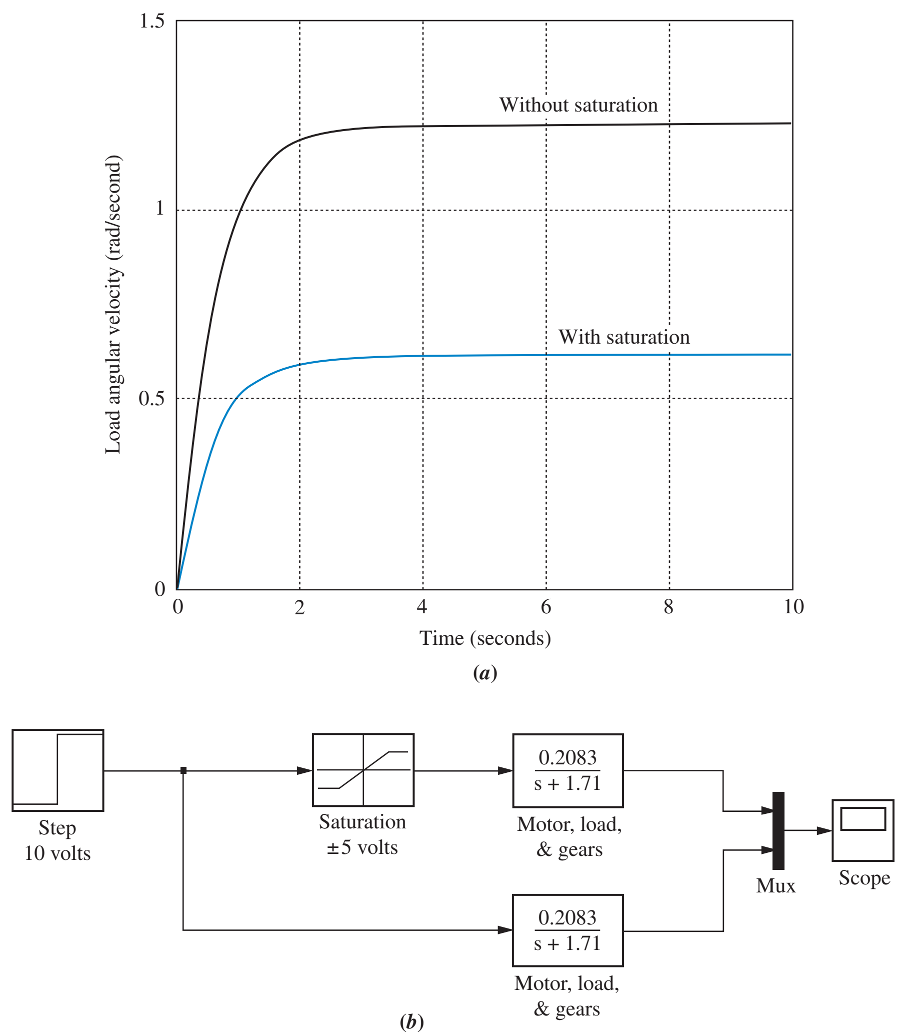
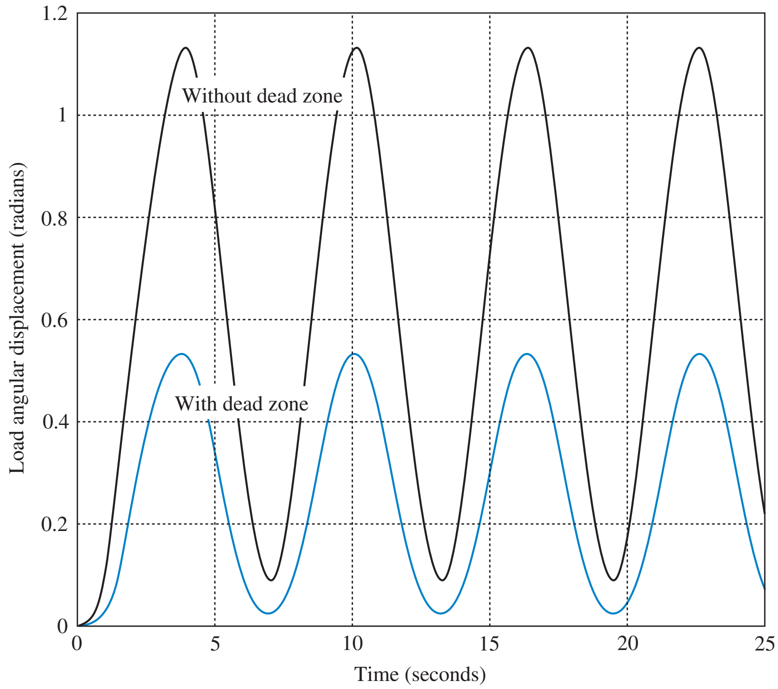
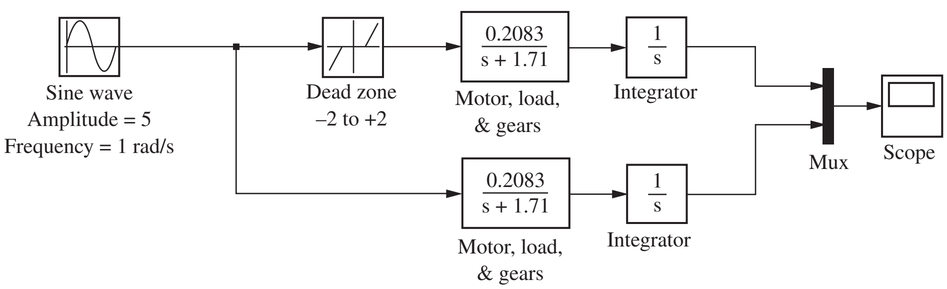


FIGURE 4.29 **a.** Effect of amplifier saturation on load angular velocity response; **b.** Simulink block diagram



(*a*)



(*b*)

FIGURE 4.30 **a.** Effect of dead zone on load angular displacement response; **b.** Simulink block diagram

The effect of dead zone on the output shaft driven by a motor and gears is shown in Figure 4.30. Here we once again assume the motor, load, and gears from Antenna Control Case Study of Chapter 2. Dead zone is present when the motor cannot respond to small voltages. The motor input is a sinusoidal waveform chosen to allow us to see the effects of dead zone vividly. The response begins when the input voltage to the motor exceeds a threshold. We notice a lower amplitude when dead zone is present.

The effect of backlash on the output shaft driven by a motor and gears is shown in Figure 4.31. Again we assume the motor, load, and gears from the Antenna Control Case Study of Chapter 2. The motor input is again a sinusoidal waveform, which is chosen to allow us to see vividly the effects of backlash in the gears driven by the motor. As the motor reverses direction, the output shaft remains stationary while the motor begins to reverse. When the gears finally connect, the output shaft itself begins to turn in the reverse direction. The resulting response is quite different from the linear response without backlash.

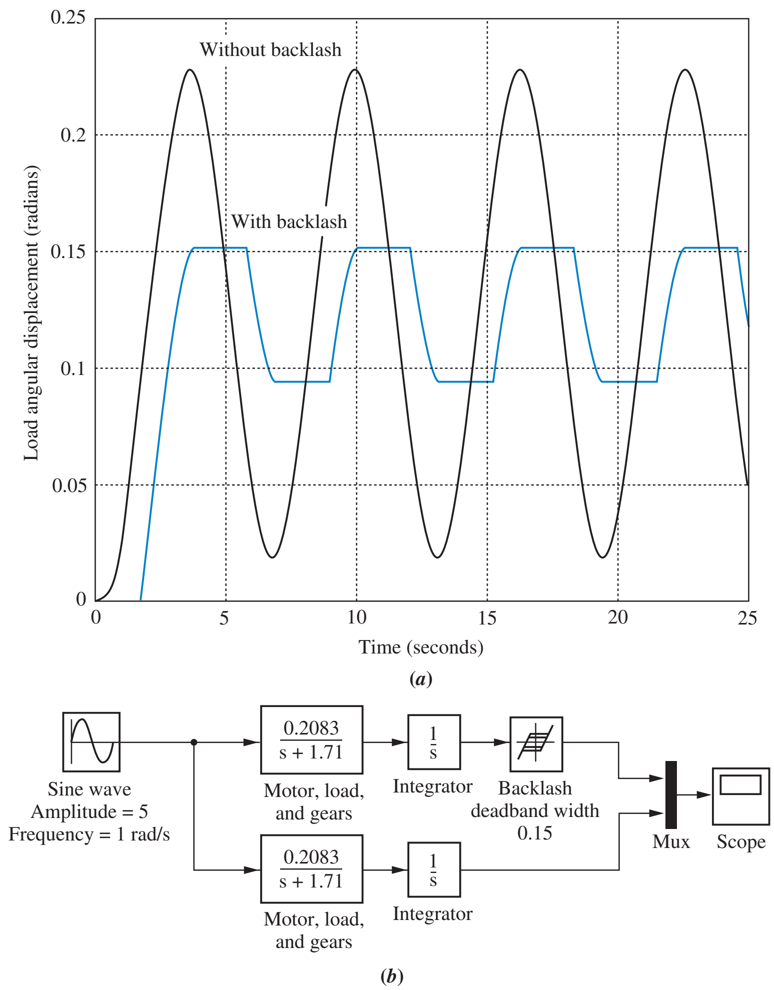


FIGURE 4.31 **a.** Effect of backlash on load angular displacement response; **b.** Simulink block diagram

Skill-Assessment Exercise 4.8

Simulink **SL**

PROBLEM: Use MATLAB's Simulink to reproduce Figure 4.31.

ANSWER: See Figure 4.31.

Now that we have seen the effects of nonlinearities on the time response, let us return to linear systems. Our coverage so far for linear systems has dealt with finding the time response by using the Laplace transform in the frequency domain. Another way to solve for the response is to use state-space techniques in the time domain. This topic is the subject of the next two sections.

4.10 Laplace Transform Solution of State Equations

State Space **SS**

In Chapter 3, systems were modeled in state space, where the state-space representation consisted of a state equation and an output equation. In this section, we use the Laplace transform to solve the state equations for the state and output vectors.

Consider the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4.92}$$

and the output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{4.93}$$

Taking the Laplace transform of both sides of the state equation yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{4.94}$$

In order to separate $\mathbf{X}(s)$, replace $s\mathbf{X}(s)$ with $s\mathbf{I}\mathbf{X}(s)$, where $\mathbf{I}$ is an $n \times n$ identity matrix, and n is the order of the system. Combining all of the $\mathbf{X}(s)$ terms, we get

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s) \tag{4.95}$$

Solving for $\mathbf{X}(s)$ by premultiplying both sides of Eq. (4.95) by $(s\mathbf{I} - \mathbf{A})^{-1}$, the final solution for $\mathbf{X}(s)$ is

$$\boxed{\begin{aligned}\mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\\ &= \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]\end{aligned}} \tag{4.96}$$

Taking the Laplace transform of the output equation yields

$$\boxed{\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)} \tag{4.97}$$

Eigenvalues and Transfer Function Poles

We saw that the poles of the transfer function determine the nature of the transient response of the system. Is there an equivalent quantity in the state-space representation that yields the same information? Section 5.8 formally defines the roots of $\det(s\mathbf{I} - \mathbf{A}) = 0$ (see the denominator of Eq. (4.96) to be *eigenvalues* of the system matrix, $\mathbf{A}$.[7] Let us show that the eigenvalues are equal to the poles of the system's transfer function. Let the output, $\mathbf{Y}(s)$, and the input, $\mathbf{U}(s)$, be scalar quantities $Y(s)$ and $U(s)$, respectively. Further, to conform to the definition of a transfer function, let $\mathbf{x}(0)$, the initial state vector, equal $\mathbf{0}$, the null vector. Substituting Eq. (4.96) into Eq. (4.97) and solving for the transfer function, $Y(s)/U(s)$, yields

$$\begin{aligned}\frac{Y(s)}{U(s)} &= \mathbf{C}\left[\frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\right]\mathbf{B} + \mathbf{D}\\ &= \frac{\mathbf{C}\operatorname{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\end{aligned} \tag{4.98}$$

The roots of the denominator of Eq. (4.98) are the poles of the system. Since the denominators of Eqs. (4.96) and (4.98) are identical, the system poles equal the eigenvalues. Hence, if a system is represented in state-space, we can find the poles from $\det(s\mathbf{I} - \mathbf{A}) = 0$. We will be more formal about these facts when we discuss stability in Chapter 6.

The following example demonstrates solving the state equations using the Laplace transform as well as finding the eigenvalues and system poles.

[7] Sometimes the symbol λ is used in place of the complex variable s when solving the state equations without using the Laplace transform. Thus, it is common to see the characteristic equation also written as $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

Example 4.11

Laplace Transform Solution; Eigenvalues and Poles

PROBLEM: Given the system represented in state space by Eqs. (4.99),

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \quad (4.99a)$$

$$y = [1 \quad 1 \quad 0]\,\mathbf{x} \quad (4.99b)$$

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad (4.99c)$$

do the following:

a. Solve the preceding state equation and obtain the output for the given exponential input.

b. Find the eigenvalues and the system poles.

SOLUTION:

a. We will solve the problem by finding the component parts of Eq. (4.96), followed by substitution into Eq. (4.97). First obtain **A** and **B** by comparing Eq. (4.99a) to Eq. (4.92). Since

$$s\mathbf{I} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \quad (4.100)$$

then

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 24 & 26 & s+9 \end{bmatrix} \quad (4.101)$$

and

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\begin{bmatrix} (s^2+9s+26) & (s+9) & 1 \\ -24 & s^2+9s & s \\ -24s & -(26s+24) & s^2 \end{bmatrix}}{s^3+9s^2+26s+24} \quad (4.102)$$

Since $\mathbf{U}(s)$ is $1/(s+1)$ (the Laplace transform for e^{-t}), $\mathbf{X}(s)$ can be calculated. Rewriting Eq. (4.96) as

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)] \quad (4.103)$$

and using **B** and $\mathbf{x}(0)$ from Eqs. (4.99a) and (4.99c), respectively, we get

$$X_1(s) = \frac{(s^3 + 10s^2 + 37s + 29)}{(s+1)(s+2)(s+3)(s+4)} \quad (4.104a)$$

$$X_2(s) = \frac{(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)} \tag{4.104b}$$

$$X_3(s) = \frac{s(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)} \tag{4.104c}$$

The output equation is found from Eq. (4.99b). Performing the indicated addition yields

$$Y(s) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = X_1(s) + X_2(s) \tag{4.105}$$

or

$$\begin{aligned} Y(s) &= \frac{(s^3 + 12s^2 + 16s + 5)}{(s+1)(s+2)(s+3)(s+4)} \\ &= \frac{-6.5}{s+2} + \frac{19}{s+3} - \frac{11.5}{s+4} \end{aligned} \tag{4.106}$$

where the pole at -1 canceled a zero at -1. Taking the inverse Laplace transform,

$$y(t) = -6.5e^{-2t} + 19e^{-3t} - 11.5e^{-4t} \tag{4.107}$$

b. The denominator of Eq. (4.102), which is $\det(s\mathbf{I} - \mathbf{A})$, is also the denominator of the system's transfer function. Thus, $\det(s\mathbf{I} - \mathbf{A}) = 0$ furnishes both the poles of the system and the eigenvalues -2, -3, and -4.

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch4sp1 in Appendix F at www.wiley.com/college/nise. You will learn how to solve state equations for the output response using the Laplace transform. Example 4.11 will be solved using MATLAB and the Symbolic Math Toolbox.

Symbolic Math
SM

Skill-Assessment Exercise 4.9

PROBLEM: Given the system represented in state space by Eqs. (4.108),

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \tag{4.108a}$$

$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \mathbf{x} \tag{4.108b}$$

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{4.108c}$$

do the following:

a. Solve for $y(t)$ using state-space and Laplace transform techniques.

b. Find the eigenvalues and the system poles.

TryIt 4.5

Use the following MATLAB and Symbolic Math Toolbox statements to solve Skill-Assessment Exercise 4.9 .

```
syms s
A=[0 2;-3 -5]; B=[0;1];
C=[1 3];X0=[2;1];
U=1/(s+1);
I=[1 0;0 1];
X=((s*I-A)^-1)*...
  (X0+B*U);
Y=C*X; Y=simplify(Y);
y=ilaplace(Y);
pretty(y)
eig(A)
```

ANSWERS:

a. $y(t) = -0.5e^{-t} - 12e^{-2t} + 17.5e^{-3t}$

b. $-2, \ -3$

The complete solution is located at www.wiley.com/college/nise.

4.11 Time Domain Solution of State Equations

State Space
SS

We now look at another technique for solving the state equations. Rather than using the Laplace transform, we solve the equations directly in the time domain using a method closely allied to the classical solution of differential equations. We will find that the final solution consists of two parts that are different from the forced and natural responses.

The solution in the time domain is given directly by

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau \\ &= \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{Bu}(\tau)d\tau\end{aligned} \tag{4.109}$$

where $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ by definition, and which is called the *state-transition matrix*. Eq. (4.109) is derived in Appendix I located at www.wiley.com/college/nise. Readers who are not familiar with this equation or who may want to refresh their memory should consult Appendix I before proceeding.

Notice that the first term on the right-hand side of the equation is the response due to the initial state vector, $\mathbf{x}(0)$. Notice also that it is the only term dependent on the initial state vector and not the input. We call this part of the response the *zero-input response*, since it is the total response if the input is zero. The second term, called the *convolution integral*, is dependent only on the input, $\mathbf{u}$, and the input matrix, $\mathbf{B}$, not the initial state vector. We call this part of the response the *zero-state response*, since it is the total response if the initial state vector is zero. Thus, there is a partitioning of the response different from the forced/natural response we have seen when solving differential equations. In differential equations, the arbitrary constants of the natural response are evaluated based on the initial conditions and the initial values of the forced response and its derivatives. Thus, the natural response's amplitudes are a function of the initial conditions of the output and the input. In Eq. (4.109), the zero-input response is not dependent on the initial values of the input and its derivatives. It is dependent only on the initial conditions of the state vector. The next example vividly shows the difference in partitioning. Pay close attention to the fact that in the final result the zero-state response contains not only the forced solution but also pieces of what we previously called the natural response. We will see in the solution that the natural response is distributed through the zero-input response and the zero-state response.

Before proceeding with the example, let us examine the form the elements of $\mathbf{\Phi}(t)$ take for linear, time-invariant systems. The first term of Eq. (4.96), the Laplace transform of the response for unforced systems, is the transform of $\Phi(t)\mathbf{x}(0)$, the zero-input response from Eq. (4.109). Thus, for the unforced system

$$\mathscr{L}[\mathbf{x}(t)] = \mathscr{L}[\mathbf{\Phi}(t)\mathbf{x}(0)] = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{4.110}$$

from which we can see that $(s\mathbf{I} - \mathbf{A})^{-1}$ *is the Laplace transform of the state-transition matrix*, $\mathbf{\Phi}(t)$. We have already seen that the denominator of $(s\mathbf{I} - \mathbf{A})^{-1}$ is a polynomial in s whose roots are the system poles. This polynomial is found from the equation $\det(s\mathbf{I}-\mathbf{A})=0$. Since

$$\boxed{\mathscr{L}^{-1}[(s\mathbf{I}-\mathbf{A})^{-1}]=\mathscr{L}^{-1}\left[\frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}\right]=\mathbf{\Phi}(t)} \tag{4.111}$$

each term of $\Phi(t)$ would be the sum of exponentials generated by the system's poles.

Let us summarize the concepts with two numerical examples. The first example solves the state equations directly in the time domain. The second example uses the Laplace transform to solve for the state-transition matrix by finding the inverse Laplace transform of $(s\mathbf{I}-\mathbf{A})^{-1}$.

Example 4.12

Time Domain Solution

PROBLEM: For the state equation and initial state vector shown in Eqs. (4.112), where $u(t)$ is a unit step, find the state-transition matrix and then solve for $\mathbf{x}(t)$.

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}0 & 1\\ -8 & -6\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u(t) \tag{4.112a}$$

$$\mathbf{x}(0)=\begin{bmatrix}1\\ 0\end{bmatrix} \tag{4.112b}$$

SOLUTION: Since the state equation is in the form

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t) \tag{4.113}$$

find the eigenvalues using $\det(s\mathbf{I}-\mathbf{A})=0$. Hence, $s^2+6s+8=0$, from which $s_1=-2$ and $s_2=-4$. Since each term of the state-transition matrix is the sum of responses generated by the poles (eigenvalues), we assume a state-transition matrix of the form

$$\mathbf{\Phi}(t)=\begin{bmatrix}(K_1e^{-2t}+K_2e^{-4t}) & (K_3e^{-2t}+K_4e^{-4t})\\ (K_5e^{-2t}+K_6e^{-4t}) & (K_7e^{-2t}+K_8e^{-4t})\end{bmatrix} \tag{4.114}$$

In order to find the values of the constants, we make use of the properties of the state-transition matrix derived in Appendix J located at www.wiley.com/college/nise.

Since

$$\mathbf{\Phi}(0)=\mathbf{I} \tag{4.115}$$

then

$$K_1+K_2=1 \tag{4.116a}$$

$$K_3+K_4=0 \tag{4.116b}$$

$$K_5+K_6=0 \tag{4.116c}$$

$$K_7+K_8=1 \tag{4.116d}$$

And, since

$$\dot{\mathbf{\Phi}}(0) = \mathbf{A} \tag{4.117}$$

then

$$-2K_1 - 4K_2 = 0 \tag{4.118a}$$

$$-2K_3 - 4K_4 = 1 \tag{4.118b}$$

$$-2K_5 - 4K_6 = -8 \tag{4.118c}$$

$$-2K_7 - 4K_8 = -6 \tag{4.118d}$$

The constants are solved by taking two simultaneous equations four times. For example, Eq. (4.116a) can be solved simultaneously with Eq. (4.118*a*) to yield the values of K_1 and K_2. Proceeding similarly, all of the constants can be found. Therefore,

$$\mathbf{\Phi}(t) = \begin{bmatrix} (2e^{-2t} - e^{-4t}) & \left(\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}\right) \\ (-4e^{-2t} + 4e^{-4t}) & (-e^{-2t} + 2e^{-4t}) \end{bmatrix} \tag{4.119}$$

Also,

$$\mathbf{\Phi}(t-\tau)\mathbf{B} = \begin{bmatrix} \left(\frac{1}{2}e^{-2(t-\tau)} - \frac{1}{2}e^{-4(t-\tau)}\right) \\ \left(-e^{-2(t-\tau)} + 2e^{-4(t-\tau)}\right) \end{bmatrix} \tag{4.120}$$

Hence, the first term of Eq. (4.109) is

$$\mathbf{\Phi}(t)\mathbf{x}(0) = \begin{bmatrix} (2e^{-2t} - e^{-4t}) \\ (-4e^{-2t} + 4e^{-4t}) \end{bmatrix} \tag{4.121}$$

The last term of Eq. (4.109) is

$$\begin{aligned} \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau &= \begin{bmatrix} \frac{1}{2}e^{-2t}\int_0^t e^{2\tau}d\tau - \frac{1}{2}e^{-4t}\int_0^t e^{4\tau}d\tau \\ -e^{-2t}\int_0^t e^{2\tau}d\tau + 2e^{-4t}\int_0^t e^{4\tau}d\tau \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{8} - \frac{1}{4}e^{-2t} + \frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix} \end{aligned} \tag{4.122}$$

Notice, as promised, that Eq. (4.122), the zero-state response, contains not only the forced response, 1/8, but also terms of the form Ae^{-2t} and Be^{-4t} that are part of what we previously called the natural response. However, the coefficients, A and B, are not dependent on the initial conditions.

The final result is found by adding Eqs. (4.121) and (4.122). Hence,

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau = \begin{bmatrix} \frac{1}{8} + \frac{7}{4}e^{-2t} - \frac{7}{8}e^{-4t} \\ -\frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t} \end{bmatrix} \tag{4.123}$$

Example 4.13

State-Transition Matrix via Laplace Transform

PROBLEM: Find the state-transition matrix of Example 4.12, using $(s\mathbf{I} - \mathbf{A})^{-1}$.

SOLUTION: We use the fact that $\mathbf{\Phi}(t)$ is the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1}$. Thus, first find $(s\mathbf{I} - \mathbf{A})$ as

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 \\ 8 & (s+6) \end{bmatrix} \tag{4.124}$$

from which

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix}}{s^2+6s+8} = \begin{bmatrix} \dfrac{s+6}{s^2+6s+8} & \dfrac{1}{s^2+6s+8} \\ \dfrac{-8}{s^2+6s+8} & \dfrac{s}{s^2+6s+8} \end{bmatrix} \tag{4.125}$$

Expanding each term in the matrix on the right by partial fractions yields

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \left(\dfrac{2}{s+2} - \dfrac{1}{s+4}\right) & \left(\dfrac{1/2}{s+2} - \dfrac{1/2}{s+4}\right) \\ \left(\dfrac{-4}{s+2} + \dfrac{4}{s+4}\right) & \left(\dfrac{-1}{s+2} + \dfrac{2}{s+4}\right) \end{bmatrix} \tag{4.126}$$

Finally, taking the inverse Laplace transform of each term, we obtain

$$\mathbf{\Phi}(t) = \begin{bmatrix} (2e^{-2t} - e^{-4t}) & \left(\dfrac{1}{2}e^{-2t} - \dfrac{1}{2}e^{-4t}\right) \\ (-4e^{-2t} + 4e^{-4t}) & (-e^{-2t} + 2e^{-4t}) \end{bmatrix} \tag{4.127}$$

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch4sp2 in Appendix F at www.wiley.com/college/nise. You will learn how to solve state equations for the output response using the convolution integral. Examples 4.12 and 4.13 will be solved using MATLAB and the Symbolic Math Toolbox.

Symbolic Math
SM

Systems represented in state space can be simulated on the digital computer. Programs such as MATLAB can be used for this purpose. Alternately, the user can write specialized programs, as discussed in Appendix H.1 at www.wiley.com/college/nise.

Students who are using MATLAB should now run ch4p3 in Appendix B. This exercise uses MATLAB to simulate the step response of systems represented in state space. In addition to generating the step response, you will learn how to specify the range on the time axis for the plot.

MATLAB
ML

Skill-Assessment Exercise 4.10

PROBLEM: Given the system represented in state space by Eqs. (4.128a):

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ -2 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \tag{4.128a}$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix}\mathbf{x} \tag{4.128b}$$

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{4.128c}$$

do the following:

a. Solve for the state-transition matrix.

b. Solve for the state vector using the convolution integral.

c. Find the output, $y(t)$.

ANSWERS:

a. $\boldsymbol{\Phi}(t) = \begin{bmatrix} \left(\frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}\right) & \left(\frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t}\right) \\ \left(-\frac{2}{3}e^{-t} + \frac{2}{3}e^{-4t}\right) & \left(-\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t}\right) \end{bmatrix}$

b. $\mathbf{x}(\mathrm{t}) = \begin{bmatrix} \left(\frac{10}{3}e^{-t} - e^{-2t} - \frac{4}{3}e^{-4t}\right) \\ \left(-\frac{5}{3}e^{-t} + e^{-2t} + \frac{8}{3}e^{-4t}\right) \end{bmatrix}$

c. $y(t) = 5e^{-t} - e^{-2t}$

The complete solution is located at www.wiley.com/college/nise.

Case Studies

Antenna Control: Open-Loop Response

In this chapter, we have made use of the transfer functions derived in Chapter 2 and the state equations derived in Chapter 3 to obtain the output response of an open-loop system. We also showed the importance of the poles of a system in determining the transient response. The following case study uses these concepts to analyze an open-loop portion of the antenna azimuth position control system. The open-loop function that we will deal with consists of a power amplifier and motor with load.

PROBLEM: For the schematic of the azimuth position control system shown on the front endpapers, Configuration 1, assume an open-loop system (feedback path disconnected).

State Space
SS
MATLAB
ML

a. Predict, by inspection, the form of the open-loop angular velocity response of the load to a step-voltage input to the power amplifier.

b. Find the damping ratio and natural frequency of the open-loop system.

c. Derive the complete analytical expression for the open-loop angular velocity response of the load to a step-voltage input to the power amplifier, using transfer functions.

d. Obtain the open-loop state and output equations.

e. Use MATLAB to obtain a plot of the open-loop angular velocity response to a step-voltage input.

SOLUTION: The transfer functions of the power amplifier, motor, and load as shown on the front endpapers, Configuration 1, were discussed in the Chapter 2 case study. The two subsystems are shown interconnected in Figure 4.32(a). Differentiating the angular position of the motor and load output by multiplying by s, we obtain the output angular velocity, ω_o, as shown in Figure 4.32(a). The equivalent transfer function representing the three blocks in Figure 4.32(a) is the product of the individual transfer functions and is shown in Figure 4.32(b).[8]

a. Using the transfer function shown in Figure 4.32(b), we can predict the nature of the step response. The step response consists of the steady-state response generated by the step input and the transient response, which is the sum of two exponentials generated by each pole of the transfer function. Hence, the form of the response is

$$\omega_o(t) = A + Be^{-100t} + Ce^{-1.71t} \tag{4.129}$$

b. The damping ratio and natural frequency of the open-loop system can be found by expanding the denominator of the transfer function. Since the open-loop transfer function is

$$G(s) = \frac{20.83}{s^2 + 101.71s + 171} \tag{4.130}$$

$\omega_n = \sqrt{171} = 13.08$, and $\zeta = 3.89$ (overdamped).

c. In order to derive the angular velocity response to a step input, we multiply the transfer function of Eq. (4.130) by a step input, $1/s$, and obtain

$$\omega_o(s) = \frac{20.83}{s(s+100)(s+1.71)} \tag{4.131}$$

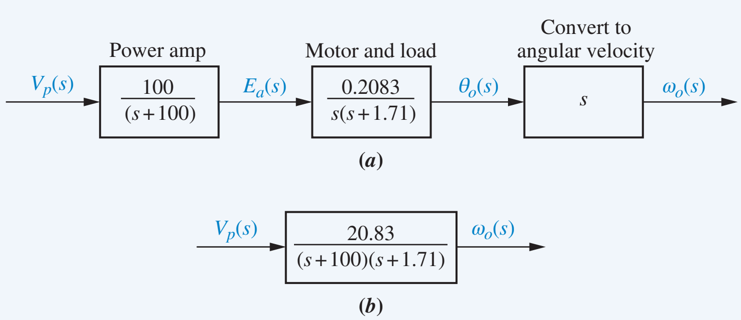


FIGURE 4.32 Antenna azimuth position control system for angular velocity: **a.** forward path; **b.** equivalent forward path

[8] This product relationship will be derived in Chapter 5.

Expanding into partial fractions, we get

$$\omega_o(s) = \frac{0.122}{s} + \frac{2.12 \times 10^{-3}}{s+100} - \frac{0.124}{s+1.71} \tag{4.132}$$

Transforming to the time domain yields

$$\omega_o(t) = 0.122 + (2.12 \times 10^{-3})e^{-100t} - 0.124e^{-1.71t} \tag{4.133}$$

State Space
SS

d. First convert the transfer function into the state-space representation. Using Eq. (4.130), we have

$$\frac{\omega_o(s)}{V_p(s)} = \frac{20.83}{s^2 + 101.71s + 171} \tag{4.134}$$

Cross-multiplying and taking the inverse Laplace transform with zero initial conditions, we have

$$\ddot{\omega}_o + 101.71\dot{\omega}_o + 171\omega_o = 20.83v_p \tag{4.135}$$

Defining the phase variables as

$$x_1 = \omega_o \tag{4.136a}$$

$$x_2 = \dot{\omega}_o \tag{4.136b}$$

and using Eq. (4.135), the state equations are written as

$$\dot{x}_1 = x_2 \tag{4.137a}$$

$$\dot{x}_2 = -171x_1 - 101.71x_2 + 20.83v_p \tag{4.137b}$$

where $v_p = 1$, a unit step. Since $x_1 = \omega_o$ is the output, the output equation is

$$y = x_1 \tag{4.138}$$

Equations (4.137) and (4.138) can be programmed to obtain the step response using MATLAB or alternative methods described in Appendix H.1 at www.wiley.com/college/nise.

MATLAB
ML

e. Students who are using MATLAB should now run ch4p4 in Appendix B. This exercise uses MATLAB to plot the step response.

CHALLENGE: You are now given a problem to test your knowledge of this chapter's objectives. Refer to the antenna azimuth position control system shown on the front endpapers, Configuration 2. Assume an open-loop system (feedback path disconnected) and do the following:

a. Predict the open-loop angular velocity response of the power amplifier, motor, and load to a step voltage at the input to the power amplifier.

b. Find the damping ratio and natural frequency of the open-loop system.

c. Derive the open-loop angular velocity response of the power amplifier, motor, and load to a step-voltage input using transfer functions.

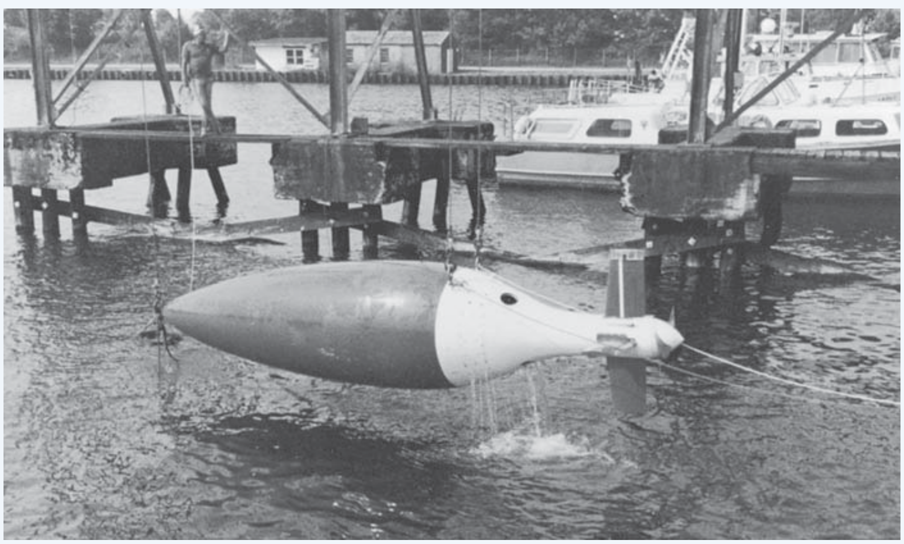
d. Obtain the open-loop state and output equations.

e. Use MATLAB to obtain a plot of the open-loop angular velocity response to a step-voltage input.

State Space
SS
MATLAB
ML

Unmanned Free-Swimming Submersible Vehicle: Open-Loop Pitch Response

An Unmanned Free-Swimming Submersible (UFSS) vehicle is shown in Figure 4.33. The depth of the vehicle is controlled as follows. During forward motion, an elevator surface on the vehicle is deflected by a selected amount. This deflection causes the vehicle to rotate about the pitch axis. The pitch of the vehicle creates a vertical force that causes the vehicle to submerge or rise. The pitch control system for the vehicle is used here and in subsequent chapters as a case study to demonstrate the covered concepts. The block diagram for the pitch control system is shown in Figure 4.34 and on the back endpapers for future reference (*Johnson*, *1980*). In this case study, we investigate the time response of the vehicle dynamics that relate the pitch angle output to the elevator deflection input.



Courtesy of the Naval Research Laboratory.

FIGURE 4.33 Unmanned Free-Swimming Submersible (UFSS) vehicle

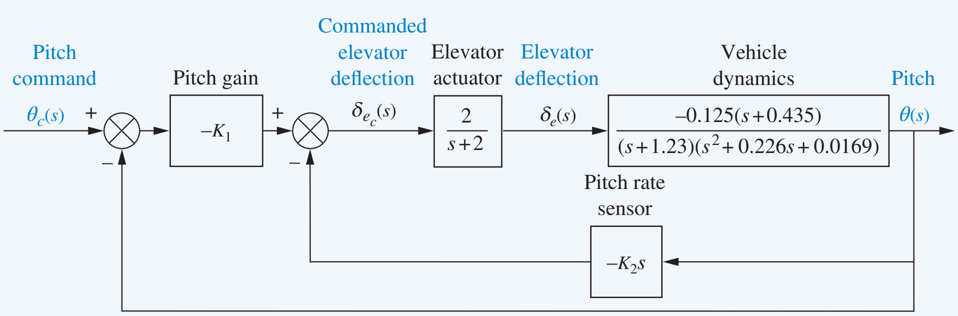


FIGURE 4.34 Pitch control loop for the UFSS vehicle

PROBLEM: The transfer function relating pitch angle, $\theta(s)$, to elevator surface angle, $\delta_e(s)$, for the UFSS vehicle is

$$\frac{\theta(s)}{\delta_e(s)} = \frac{-0.125(s+0.435)}{(s+1.23)(s^2+0.226s+0.0169)} \tag{4.139}$$

a. Using only the second-order poles shown in the transfer function, predict percent overshoot, rise time, peak time, and settling time.

b. Using Laplace transforms, find the analytical expression for the response of the pitch angle to a step input in elevator surface deflection.

c. Evaluate the effect of the additional pole and zero on the validity of the second-order approximation.

d. Plot the step response of the vehicle dynamics and verify your conclusions found in (c). An animation PowerPoint presentation (PPT) demonstrating this system is available for instructors at www.wiley.com/college/nise. See *UFSS Vehicle*.

SOLUTION:

a. Using the polynomial $s^2 + 0.226s + 0.0169$, we find that $\omega_n^2 = 0.0169$ and $2\zeta\omega_n = 0.226$. Thus, $\omega_n = 0.13$ rad/s and $\zeta = 0.869$. Hence, $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}}100 = 0.399\%$. From Figure 4.16, $\omega_n T_r = 2.75$, or $T_r = 21.2$ s. To find peak time, we use $T_p = \pi/\omega_n\sqrt{1-\zeta^2} = 48.9$ s. Finally, settling time is $T_s = 4/\zeta\omega_n = 35.4$ s.

b. In order to display a positive final value in Part **d**. we find the response of the system to a negative unit step, compensating for the negative sign in the transfer function. Using partial-fraction expansion, the Laplace transform of the response, $\theta(s)$, is

$$\begin{aligned}\theta(s) &= \frac{0.125(s+0.435)}{s(s+1.23)(s^2+0.226s+0.0169)} \\ &= 2.616\frac{1}{s} + 0.0645\frac{1}{s+1.23} \\ &\quad - \frac{2.68(s+0.113)+3.478\sqrt{0.00413}}{(s+0.113)^2+0.00413}\end{aligned} \tag{4.140}$$

Taking the inverse Laplace transform,

$$\begin{aligned}\theta(t) &= 2.616 + 0.0645e^{-1.23t} \\ &\quad - e^{-0.113t}(2.68\cos 0.0643t + 3.478\sin 0.0643t) \\ &= 2.616 + 0.0645e^{-1.23t} - 4.39e^{-0.113t}\cos(0.0643t + 52.38^\circ)\end{aligned} \tag{4.141}$$

c. Looking at the relative amplitudes between the coefficient of the $e^{-1.23t}$ term and the cosine term in Eq. (4.165), we see that there is pole-zero cancellation between the pole at -1.23 and the zero at -0.435. Further, the pole at -1.23 is more than five times farther from the $j\omega$ axis than the second-order dominant poles at $-0.113 \pm j0.0643$. We conclude that the response will be close to that predicted.

d. Plotting Eq. (4.141) or using a computer simulation, we obtain the step response shown in Figure 4.35. We indeed see a response close to that predicted.

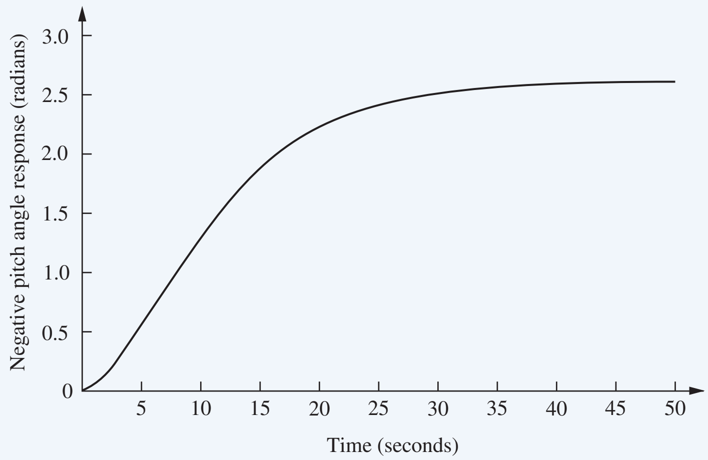


FIGURE 4.35 Negative step response of pitch control for UFSS vehicle

MATLAB ML

Students who are using MATLAB should now run ch4p5 in Appendix B. This exercise uses MATLAB to find ζ, ω, T_s, T_p, and T_r and plot a step response. Table lookup is used to find T_r. The exercise applies the concepts to the problem above.

CHALLENGE: You are now given a problem to test your knowledge of this chapter's objectives. This problem uses the same principles that were applied to the Unmanned Free-Swimming Submersible vehicle: Ships at sea undergo motion about their roll axis, as shown in Figure 4.36. Fins called *stabilizers* are used to reduce this rolling motion. The stabilizers can be positioned by a closed-loop roll control system that consists of components, such as fin actuators and sensors, as well as the ship's roll dynamics.

Assume the roll dynamics, which relates the roll-angle output, $\theta(s)$, to a disturbance-torque input, $T_D(s)$, is

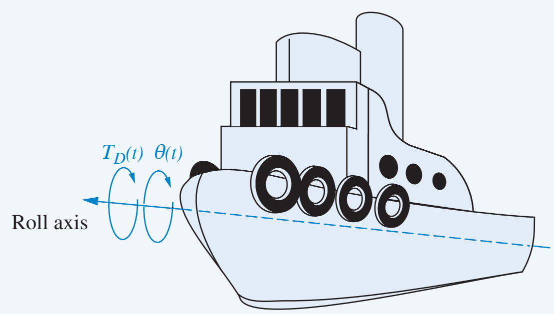


FIGURE 4.36 A ship at sea, showing roll axis

$$\frac{\theta(s)}{T_D(s)} = \frac{2.25}{(s^2 + 0.5s + 2.25)} \tag{4.142}$$

Do the following:

a. Find the natural frequency, damping ratio, peak time, settling time, rise time, and percent overshoot.

b. Find the analytical expression for the output response to a unit step input in voltage.

c. Use MATLAB to solve a and b and to plot the response found in b.

MATLAB ML

Summary

In this chapter, we took the system models developed in Chapters 2 and 3 and found the output response for a given input, usually a step. The step response yields a clear picture of the system's transient response. We performed this analysis for two types of systems, *first order* and *second order*, which are representative of many physical systems. We then formalized our findings and arrived at numerical specifications describing the responses.

For first-order systems having a single pole on the real axis, the specification of transient response that we derived was the *time constant*, which is the reciprocal of the real-axis pole location. This specification gives us an indication of the speed of the transient response. In particular, the time constant is the time for the step response to reach 63% of its final value.

Second-order systems are more complex. Depending on the values of system components, a second-order system can exhibit four kinds of behavior:

1. Overdamped
2. Underdamped
3. Undamped
4. Critically damped

We found that the poles of the input generate the forced response, whereas the system poles generate the transient response. If the system poles are real, the system exhibits *overdamped* behavior. These exponential responses have time constants equal to the reciprocals of the pole locations. Purely imaginary poles yield *undamped* sinusoidal oscillations whose radian frequency is equal to the magnitude of the imaginary pole. Systems with complex poles display *underdamped* responses. The real part of the complex pole dictates the exponential decay envelope, and the imaginary part dictates the sinusoidal radian frequency. The exponential decay envelope has a time constant equal to the reciprocal of the real part of the pole, and the sinusoid has a radian frequency equal to the imaginary part of the pole.

For all second-order cases, we developed specifications called the *damping ratio*, ζ, and *natural frequency*, ω_n. The damping ratio gives us an idea about the nature of the transient response and how much overshoot and oscillation it undergoes, regardless of time scaling. The natural frequency gives an indication of the speed of the response.

We found that the value of ζ determines the form of the second-order natural response:

- If $\zeta = 0$, the response is undamped.
- If $\zeta < 1$, the response is underdamped.
- If $\zeta = 1$, the response is critically damped.
- If $\zeta > 1$, the response is overdamped.

The natural frequency is the frequency of oscillation if all damping is removed. It acts as a scaling factor for the response, as can be seen from Eq. (4.28), in which the independent variable can be considered to be $\omega_n t$.

For the underdamped case we defined several transient response specifications, including these:

- Percent overshoot, $\%OS$
- Peak time, T_p
- Settling time, T_s
- Rise time, T_r

The peak time is inversely proportional to the imaginary part of the complex pole. Thus, horizontal lines on the s-plane are lines of constant peak time. Percent overshoot is a function of only the damping ratio. Consequently, radial lines are lines of constant percent overshoot. Finally, settling time is inversely proportional to the real part of the complex pole. Hence, vertical lines on the s-plane are lines of constant settling time.

We found that peak time, percent overshoot, and settling time are related to pole location. Thus, we can design transient responses by relating a desired response to a pole location and then relating that pole location to a transfer function and the system's components.

The effects of nonlinearities, such as saturation, dead zone, and backlash, were explored using MATLAB's Simulink.

In this chapter, we also evaluated the time response using the state-space approach. The response found in this way was separated into the *zero-input response*, and the *zero-state response*, whereas the frequency response method yielded a total response divided into *natural response* and *forced response* components.

In the next chapter we will use the transient response specifications developed here to analyze and design systems that consist of the interconnection of multiple subsystems. We will see how to reduce these systems to a single transfer function in order to apply the concepts developed in Chapter 4.

Review Questions

1. Name the performance specification for first-order systems.
2. What does the performance specification for a first-order system tell us?
3. In a system with an input and an output, what poles generate the steady-state response?
4. In a system with an input and an output, what poles generate the transient response?
5. The imaginary part of a pole generates what part of a response?
6. The real part of a pole generates what part of a response?
7. What is the difference between the natural frequency and the damped frequency of oscillation?
8. If a pole is moved with a constant imaginary part, what will the responses have in common?
9. If a pole is moved with a constant real part, what will the responses have in common?
10. If a pole is moved along a radial line extending from the origin, what will the responses have in common?
11. List five specifications for a second-order underdamped system.
12. For Question 11 how many specifications completely determine the response?
13. What pole locations characterize (1) the underdamped system, (2) the overdamped system, and (3) the critically damped system?
14. Name two conditions under which the response generated by a pole can be neglected.
15. How can you justify pole-zero cancellation?
16. Does the solution of the state equation yield the output response of the system? Explain. State Space SS
17. What is the relationship between $(s\mathbf{I} - \mathbf{A})$, which appeared during the Laplace transformation solution of the state equations, and the state-transition matrix, which appeared during the classical solution of the state equation? State Space SS
18. Name a major advantage of using time-domain techniques for the solution of the response. State Space SS
19. Name a major advantage of using frequency-domain techniques for the solution of the response.