in Figure 2.32(b), from which the equation of motion is

$$(J_e s^2 + D_e s)\theta_1(s) = T_1(s) \tag{2.142}$$

where

$$J_e = J_1 + (J_2 + J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5)\left(\frac{N_1 N_3}{N_2 N_4}\right)^2$$

and

$$D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$

From Eq. (2.142), the transfer function is

$$G(s) = \frac{\theta_1(s)}{T_1(s)} = \frac{1}{J_e s^2 + D_e s} \tag{2.143}$$

as shown in Figure 2.32(c).

Skill-Assessment Exercise 2.10

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system with gears shown in Figure 2.33.

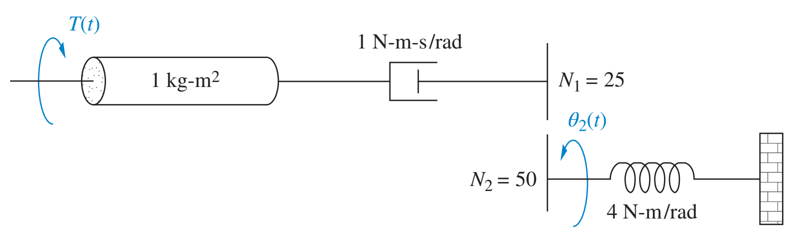


FIGURE 2.33 Rotational mechanical system with gears for Skill-Assessment Exercise 2.10

ANSWER: $G(s) = \dfrac{1/2}{s^2 + s + 1}$

The complete solution is at www.wiley.com/college/nise.

2.8 Electromechanical System Transfer Functions

In the last section we talked about rotational systems with gears, which completed our discussion of purely mechanical systems. Now, we move to systems that are hybrids of electrical and mechanical variables, the *electromechanical systems*. We have seen one application of an electromechanical system in Chapter 1, the antenna azimuth position

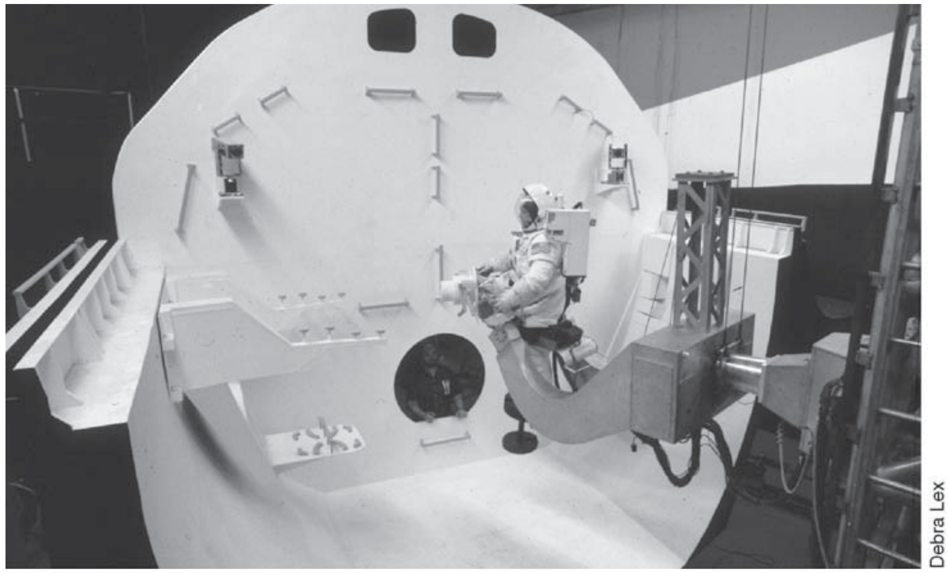


FIGURE 2.34 NASA flight simulator robot arm with electromechanical control system components

control system. Other applications for systems with electromechanical components are robot controls, sun and star trackers, and computer tape and disk-drive position controls. An example of a control system that uses electromechanical components is shown in Figure 2.34.

A motor is an electromechanical component that yields a displacement output for a voltage input, that is, a mechanical output generated by an electrical input. We will derive the transfer function for one particular kind of electromechanical system, the armature-controlled dc servomotor (*Mablekos*, *1980*). The motor's schematic is shown in Figure 2.35(*a*), and the transfer function we will derive appears in Figure 2.35(*b*).

In Figure 2.35(*a*) a magnetic field is developed by stationary permanent magnets or a stationary electromagnet called the *fixed field*. A rotating circuit called the *armature*, through which current $i_a(t)$ flows, passes through this magnetic field at right angles and feels a force, $F = Bli_a(t)$, where B is the magnetic field strength and l is the length of the conductor. The resulting torque turns the *rotor*, the rotating member of the motor.

There is another phenomenon that occurs in the motor: A conductor moving at right angles to a magnetic field generates a voltage at the terminals of the conductor equal to

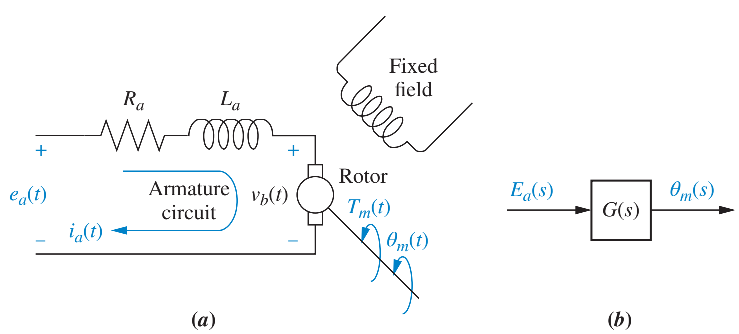


FIGURE 2.35 DC motor: **a.** schematic;[12] **b.** block diagram

[12] See Appendix I at www.wiley.com/college/nise for a derivation of this schematic and its parameters.

$e = Blv$, where e is the voltage and v is the velocity of the conductor normal to the magnetic field. Since the current-carrying armature is rotating in a magnetic field, its voltage is proportional to speed. Thus,

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt} \tag{2.144}$$

We call $v_b(t)$ the *back electromotive force* (*back emf*); K_b is a constant of proportionality called the back emf constant; and $d\theta_m(t)/dt = \omega_m(t)$ is the angular velocity of the motor. Taking the Laplace transform, we get

$$V_b(s) = K_b s\theta_m(s) \tag{2.145}$$

The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$, is found by writing a loop equation around the Laplace transformed armature circuit (see Figure 3.5(a)):

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s) \tag{2.146}$$

The torque developed by the motor is proportional to the armature current; thus,

$$T_m(s) = K_t I_a(s) \tag{2.147}$$

where T_m is the torque developed by the motor, and K_t is a constant of proportionality, called the motor torque constant, which depends on the motor and magnetic field characteristics. In a consistent set of units, the value of K_t is equal to the value of K_b. Rearranging Eq. (2.147) yields

$$I_a(s) = \frac{1}{K_t} T_m(s) \tag{2.148}$$

To find the transfer function of the motor, we first substitute Eqs. (2.145) and (2.148) into (2.146), yielding

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \tag{2.149}$$

Now we must find $T_m(s)$ in terms of $\theta_m(s)$ if we are to separate the input and output variables and obtain the transfer function, $\theta_m(s)/E_a(s)$.

Figure 2.36 shows a typical equivalent mechanical loading on a motor. J_m is the equivalent inertia at the armature and includes both the armature inertia and, as we will see later, the load inertia reflected to the armature. D_m is the equivalent viscous damping at the armature and includes both the armature viscous damping and, as we will see later, the load viscous damping reflected to the armature. From Figure 2.36,

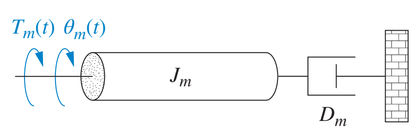


FIGURE 2.36 Typical equivalent mechanical loading on a motor

$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s) \tag{2.150}$$

Substituting Eq. (2.150) into Eq. (2.149) yields

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \tag{2.151}$$

If we assume that the armature inductance, L_a, is small compared to the armature resistance, R_a, which is usual for a dc motor, Eq. (2.151) becomes

$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b\right] s\theta_m(s) = E_a(s) \tag{2.152}$$

After simplification, the desired transfer function, $\theta_m(s)/E_a(s)$, is found to be

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]} \tag{2.153}$$ [13]

Even though the form of Eq. (2.153) is relatively simple, namely

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s+\alpha)} \tag{2.154}$$

the reader may be concerned about how to evaluate the constants.

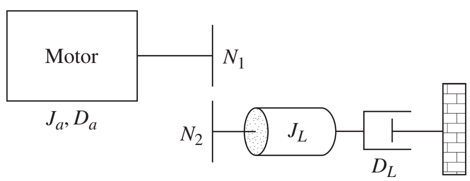


FIGURE 2.37 DC motor driving a rotational mechanical load

Let us first discuss the mechanical constants, J_m and D_m. Consider Figure 2.37, which shows a motor with inertia J_a and damping D_a at the armature driving a load consisting of inertia J_L and damping D_L. Assuming that all inertia and damping values shown are known, J_L and D_L can be reflected back to the armature as some equivalent inertia and damping to be added to J_a and D_a, respectively. Thus, the equivalent inertia, J_m, and equivalent damping, D_m, at the armature are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \quad D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 \tag{2.155}$$ [14]

Now that we have evaluated the mechanical constants, J_m and D_m, what about the electrical constants in the transfer function of Eq. (2.153)? We will show that these constants can be obtained through a *dynamometer* test of the motor, where a dynamometer measures the torque and speed of a motor under the condition of a constant applied voltage. Let us first develop the relationships that dictate the use of a dynamometer.

Substituting Eqs. (2.145) and (2.148) into Eq. (2.146), with $L_a = 0$, yields

$$\frac{R_a}{K_t}T_m(s) + K_b s\theta_m(s) = E_a(s) \tag{2.156}$$

Taking the inverse Laplace transform, we get

$$\frac{R_a}{K_t}T_m(t) + K_b\omega_m(t) = e_a(t) \tag{2.157}$$

where the inverse Laplace transform of $s\theta_m(s)$ is $d\theta_m(t)/dt$ or, alternately, $\omega_m(t)$.

If a dc voltage, e_a, is applied, the motor will turn at a constant angular velocity, ω_m, with a constant torque, T_m. Hence, dropping the functional relationship based on time from Eq. (2.157), the following relationship exists when the motor is operating at steady state with a dc voltage input:

$$\frac{R_a}{K_t}T_m + K_b\omega_m = e_a \tag{2.158}$$

[13] The units for the electrical constants are K_t = N-m-A (newton-meters/ampere), and K_b = V-s/rad (volt-seconds/radian).

[14] If the values of the mechanical constants are not known, motor constants can be determined through laboratory testing using transient response or frequency response data. The concept of transient response is covered in Chapter 4; frequency response is covered in Chapter 10.

Solving for T_m yields

$$T_m = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a}e_a \tag{2.159}$$

Equation (2.159) is a straight line, T_m vs. ω_m, and is shown in Figure 2.38. This plot is called the *torque-speed curve*. The torque axis intercept occurs when the angular velocity reaches zero. That value of torque is called the *stall torque*, T_{stall}. Thus,

$$T_{stall} = \frac{K_t}{R_a}e_a \tag{2.160}$$

The angular velocity occurring when the torque is zero is called the *no-load speed,* $\omega_{no\text{-}load}$. Thus,

$$\omega_{no\text{-}load} = \frac{e_a}{K_b} \tag{2.161}$$

The electrical constants of the motor's transfer function can now be found from Eqs. (2.160) and (2.161) as

$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} \tag{2.162}$$

and

$$K_b = \frac{e_a}{\omega_{no\text{-}load}} \tag{2.163}$$

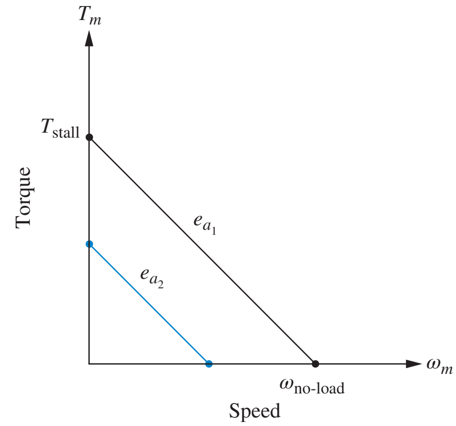


FIGURE 2.38 Torque-speed curves with an armature voltage, e_a, as a parameter

The electrical constants, K_t/R_a and K_b, can be found from a dynamometer test of the motor, which would yield T_{stall} and $\omega_{no\text{-}load}$ for a given e_a.

Example 2.23

Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(*a*) and (*b*), find the transfer function, $\theta_L(s)/E_a(s)$.

SOLUTION: Begin by finding the mechanical constants, J_m and D_m, in Eq. (2.153). From Eq. (2.155), the total inertia at the armature of the motor is

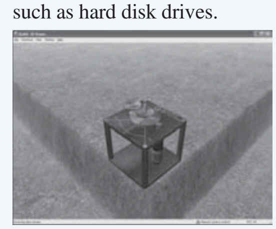
$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12 \tag{2.164}$$

and the total damping at the armature of the motor is

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10 \tag{2.165}$$

Virtual Experiment 2.2
Open-Loop Servo Motor

Put theory into practice exploring the dynamics of the Quanser Rotary Servo System modeled in LabVIEW. It is particularly important to know how a servo motor behaves when using them in high-precision applications such as hard disk drives.



Virtual experiments are found on Learning Space.

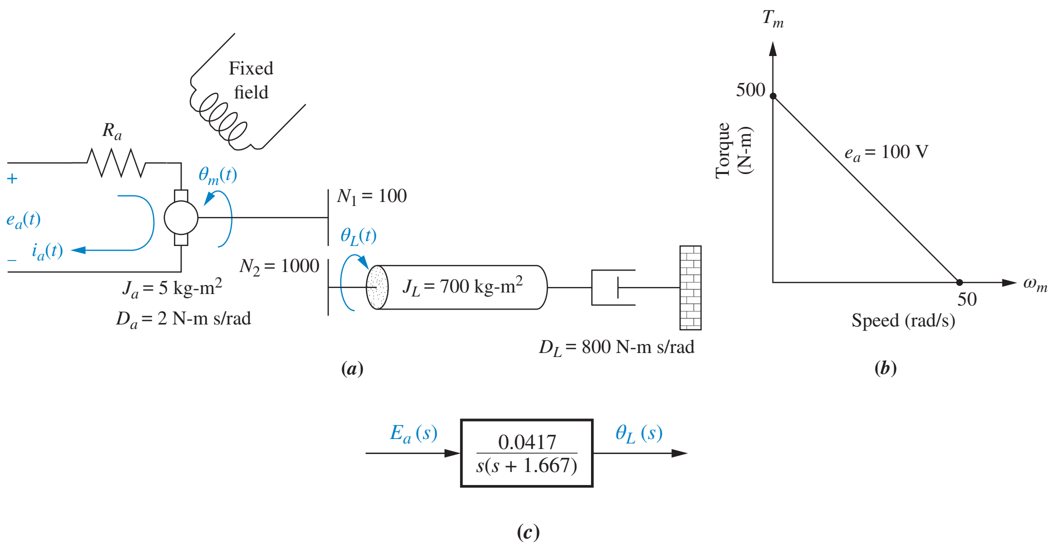


FIGURE 2.39 **a.** DC motor and load; **b.** torque-speed curve; **c.** block diagram

Now we will find the electrical constants, K_t/R_a and K_b. From the torque-speed curve of Figure 2.39(*b*),

$$T_{\text{stall}} = 500 \tag{2.166}$$

$$\omega_{\text{no-load}} = 50 \tag{2.167}$$

$$e_a = 100 \tag{2.168}$$

Hence the electrical constants are

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} = \frac{500}{100} = 5 \tag{2.169}$$

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2 \tag{2.170}$$

Substituting Eqs. (2.164), (2.165), (2.169), and (2.170) into Eq. (2.153) yield

$$\frac{\theta_m(s)}{E_a(s)} = \frac{5/12}{s\left\{s + \frac{1}{12}[10 + (5)(2)]\right\}} = \frac{0.417}{s(s+1.667)} \tag{2.171}$$

In order to find $\theta_L(s)/E_a(s)$, we use the gear ratio, $N_1/N_2 = 1/10$, and find

$$\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)} \tag{2.172}$$

as shown in Figure 2.39(*c*).