**Abstract**

We present analytical, experimental, and numerical results for the Saffman-Taylor instability for a two-phase flow in a Hele-Shaw Cell. Experimentally, we have considered few different fluid combinations water-glycerol and water-PEO (polyethylene oxide): PEO is a non-Newtonian fluid that exhibit more complex behavior such as shear thinning and elastic response. Theoretically, we have analyzed the stability of the simple (Newtonian) fluid interface and compared the predictions with the experimental results. Computationally, we have carried out Monte-Carlo type of simulations based on the so-called diffusion limited aggregation (DLA) approach. We have computed various measures of the emerging patterns, including fractal dimension for both experimental and computational results.

**Linear Stability Analysis (LSA)**

Stokes flow is a creeping flow whose transport force is smaller than the viscous force. Furthermore, the Reynolds number (ratio of inertial to viscous forces) is small (Re<<1). A Hele-Shaw flow is a specific case of it, where the flow between two parallel plates that are spaced close to each other is considered. The Saffman-Taylor instability (viscous fingering) occurs when a less viscous fluid is forced into a more viscous fluid.

The model instability we consider the fluid-fluid interface to initially be circular with some small sinusoidal perturbation characterized by a wavenumber \( m \). In addition, we include surface tension forces. At the injection point, we assume the flow satisfies Darcy’s Law. Under these modelling considerations we derive the growth rate of the perturbations as a function of \( m \), \( m \), and \( m \) respectively. The growth rate is then the mode that is observed, the number of observed fingers can be derived.

**Experiment**

- **Setup/Procedure (see Figure 1)**
  - Step 1: 4–6 posts are screwed into optical table (A) with wax paper background (B) resting on top
  - Step 2: Various sizes of spacers are placed between the plastic plates (I) for separation.
  - Step 3: A more viscous fluid (2) is poured on bottom plate and flattened out in between two plates (I).
  - Step 4: Elevated syringe with tubing (D) is connected by less viscous fluid (C) through a needle to a hole in the bottom plate.
  - Step 5: Various weights (D) placed on top of syringe.
  - Step 6: Simple smartphone video camera (A) placed on stand directly above cell.

**Diffusion Limited Aggregation**

Diffusion limited aggregation is a numerical method based on the idea of Brownian motion. The random walkers begin walking from the outer edge of a circle whose center is the initial seed. The place the walker starts from is random and the movement of the walker follows is also random— the walker moves north, south, east, or west until it reaches a previously stuck walker. The only thing a walker can do is leave the circle and if they attempt to do so, a new random movement is generated. As the aggregate of walkers grows larger, the circle expands to contain the entire aggregate. Calculating the sticking probability is based on the following method: sticking probability is linearly dependent on the number of occupied spaces around the current location \( N \), which is calculated by looking at a 1 by 1 matrix around the space the walker wants to stick to (for our simulations we mainly used 1):

\[
\lim_{t \to \infty} \frac{\text{size of aggregate}}{\text{time}} = \text{constant}
\]

In Newtonian fluids the viscosity is constant. In the non-Newtonian case we use the idea that the shear rate depends on the fluid velocity. To simulate this effect, we adjust the sticking probability function such that the probability is higher around the recently occupied spaces. In order to implement this, one has to keep track of the order in which the particles stick to the aggregate and mark the unoccupied spaces around the interface by an order value. We refer to this as the velocity number. In each addition to the cluster, the most recent stuck walker will be assigned a velocity number of one and the rest of the velocity numbers will be incremented to the order.

If the walker lands on the aggregate at a point where the velocity number is low, the probability of sticking is larger. Therefore, this local value at the point where the walker wants to stick appears in the denominator of the correction (see the probability function in the text that follows).

Furthermore, as the aggregate grows, the probability to stick on any spot on the interface gets smaller, simply because there are more spots to stick.

**Boundary Integral Methods (BIM)**

The idea of BIM is that we can approximate the solution of a PDE by first finding the solution at the boundary. This is used primarily with large domains (both interior and exterior) where the finite difference method would require too many points for practicality.

We are looking for the solution \( LP = 0 \) where \( L \) is \( \partial \). A Green’s function is any function that under the linear operator \( L \) fulfills \( L(G(x,y)) = δ(x−y) \). By the principle of Green’s identities, we have

\[
\int\int_{Ω} (L(G(x,y))u(x,y)) \, dx \, dy = \int_{∂Ω} (G_{|x|=0} \cdot u) \, ds
\]

Calculating the Green’s function is based on the following method:

\[
\text{Green’s function} = \begin{cases} \frac{1}{4\pi} \ln \left| \frac{r}{r_0} \right| & r < r_0 \\frac{1}{2\pi} \tan^{-1} \left( \frac{x_1-x}{x_2-y} \right) & r > r_0 \end{cases}
\]

where \( x \) and \( y \) are the source and observer points, respectively.

**Application of Complex Variables**

The movement of a fluid boundary within a Hele-Shaw cell can be modeled by the use of complex variables. With complex variables, a conformal map can be deduced within a zero surface tension (2D) environment. A point from the Epane can be mapped to the physical a plane. This new domain will evolve over time.

**Reference**