**CS 341: Foundations of Computer Science II**
**Prof. Marvin Nakayama**

# Homework 1 Solutions

1. Answer the questions below for the sets of strings:

$$\begin{aligned} C &= \{\varepsilon, aab, baa\}, \\ D &= \{bb, aab\}, \\ E &= \{\varepsilon\}, \\ F &= \emptyset. \end{aligned}$$

Except for parts (c) and (h), write out any sets with the elements in *string order*, i.e., shorter strings appear before longer strings, and strings of the same length are in alphabetical order. For any infinite sets, list only the first 8 elements in string order, but be sure to also indicate that the set is infinite by including 3 dots after listing the first 8 elements. For any set $S$ of strings, we define

$$\begin{aligned} S^* &= \{\, x_1 x_2 \cdots x_k \mid k \geq 0 \text{ and each } x_i \in S \,\}, \\ S^+ &= \{\, x_1 x_2 \cdots x_k \mid k \geq 1 \text{ and each } x_i \in S \,\}, \end{aligned}$$

where the concatenation of $k = 0$ strings is the empty string $\varepsilon$.

(a) What is $D \cup C$ ?

**Answer:** $D \cup C = \{\varepsilon, bb, aab, baa\}$

(b) What is $C \cup F$ ?

**Answer:** $C \cup F = \{\varepsilon, aab, baa\} = C$

(c) What is $C \times D$ ?

**Answer:** $C \times D = \{\ (\varepsilon, bb),\ (\varepsilon, aab),\ (aab, bb),\ (aab, aab),\ (baa, bb),\ (baa, aab)\ \}$

(d) What is $C \cap D$ ?

**Answer:** $C \cap D = \{aab\}$

(e) What is $D \circ C$ ?

**Answer:** $D \circ C = \{bb, aab, bbaab, bbbaa, aabaab, aabbaa\}$

(f) What is $C \circ E$ ?

**Answer:** $C \circ E = \{\varepsilon, aab, baa\}$

(g) What is $D \circ D \circ D$ ?

**Answer:** $D \circ D \circ D = \{bbbbbb,\ aabbbbb,\ bbaabbb,\ bbbbaab,\ aabaabbb,\ aabbbaab,$ $bbaabaab,\ aabaabaab\}$

(h) What is $\mathcal{P}(C)$ ?

**Answer:** $\mathcal{P}(C) = \{\ \emptyset,\ \{\varepsilon\},\ \{aab\},\ \{baa\},\ \{\varepsilon, aab\},\ \{\varepsilon, baa\},\ \{aab, baa\},\ \{\varepsilon, aab, baa\}\ \}$

(i) What is $D - C$ ?

**Answer:** $D - C = \{bb\}$

(j) What is $C^+$ ?

**Answer:** $C^+ = \{\varepsilon, aab, baa, aabaab, aabbaa, baaaab, baabaa, aabaabaab, \ldots\}$

(k) What is $F^*$ ?

**Answer:** $F^* = \{\varepsilon\}$

(l) Is $E \subseteq C$ ?

**Answer:** $E \subseteq C$ since every element of $E$ is also in $C$.

(m) Is $D \subseteq C$ ?

**Answer:** $D \not\subseteq C$ since $bb \in D$ but $bb \notin C$.

(n) For this part, we first need some definitions.

- The *reverse* of a string $w$, written $w^{\mathcal{R}}$, is the same string with the symbols written in reverse order. For example, $(cat)^{\mathcal{R}} = tac$.
- A collection of objects is *closed* under some operation if applying that operation to members of the collection returns an object still in the collection. For example, the set $\mathcal{N} = \{0, 1, 2, 3, \ldots\}$ is closed under the function $f(x) = x^2$ since the square of any nonnegative integer always results in a nonnegative integer. However, $\mathcal{N}$ is not closed under square root since, for example, $3 \in \mathcal{N}$ but $\sqrt{3} \notin \mathcal{N}$.

Which of $C$, $D$, and $E$ are closed under reversal? Explain your answers.

**Answer:**

- $C$ is closed under reversal since $\varepsilon^{\mathcal{R}} = \varepsilon \in C$, $(aab)^{\mathcal{R}} = baa \in C$, and $(baa)^{\mathcal{R}} = aab \in C$.
- $D$ is not closed under reversal since $(aab)^{\mathcal{R}} = baa \notin D$.
- $E$ is closed under reversal since $\varepsilon^{\mathcal{R}} = \varepsilon \in E$.

2. Let $w$ be a string of symbols, and let the language $T$ be defined as adding $w$ to the language $S$; i.e., $T = S \cup \{w\}$. Suppose further that $T^* = S^*$.

   (a) Is it necessarily true that $w \in S$? If this is necessarily true, give a proof. If this is not necessarily true, give a counterexample.

   **Answer:** It is not true in general that $w \in S$. For example, suppose that $w = aa$, $S = \{a\}$, and $T = \{a, aa\}$. Then note that $T = S \cup \{aa\}$, and $S^* = T^* = \{\varepsilon, a, aa, aaa, \ldots\}$, but $aa \notin S$.

   (b) Is it necessarily true that $w \in S^*$? If this is necessarily true, give a proof. If this is not necessarily true, give a counterexample.

   **Answer:** It must be the case that $w \in S^*$. Note that $w \in T$, so $w \in T^*$ since any string in $T$ is also in $T^*$ because $T \subset T^*$. But since $T^* = S^*$, we must have that $w \in S^*$.

3. For each of the following parts, give an example satisfying the given conditions. Give a brief explanation for each of your examples.

   (a) Give an example of a set $S$ of strings such that $S^* = S^+$.

   **Answer:** Let $S = \{\varepsilon, a\}$. Then $S^* = \{\varepsilon, a, aa, aaa, \ldots\}$ and $S^+ = \{\varepsilon, a, aa, aaa, \ldots\}$, so $S^* = S^+$.

   (b) Give an example of a set $S$ of strings such that $S^* \neq S^+$.

   **Answer:** Let $S = \{a\}$. Then $S^* = \{\varepsilon, a, aa, aaa, \ldots\}$ and $S^+ = \{a, aa, aaa, \ldots\}$, so $S^* \neq S^+$.

   (c) Give an example of a set $S$ of strings such that $S = S^*$.

   **Answer:** Let $S = \{\varepsilon, a, aa, aaa, \ldots\}$. Then $S^* = \{\varepsilon, a, aa, aaa, \ldots\}$, so $S = S^*$.

   (d) Give an example of a set $S$ of strings such that $S \neq S^*$.

   **Answer:** Let $S = \{a\}$. Then $S^* = \{\varepsilon, a, aa, aaa, \ldots\}$, so $S \neq S^*$.

   (e) Give an example of a set $S$ of strings such that $S^*$ is finite.

   **Answer:** Let $S = \{\varepsilon\}$. Then $S^* = \{\varepsilon\}$, so $S^*$ is finite.

4. Suppose we define a restricted version of the Java programming language in which variable names must satisfy all of the following conditions:

   - A variable name can only use Roman letters (i.e., a, b, ..., z, A, B, ..., Z) or Arabic numerals (i.e., 0, 1, 2, ..., 9); i.e., underscore is not allowed.
   - A variable name must start with a Roman letter: a, b, ..., z, A, B, ..., Z

- The length of a variable name must be no greater than 8.
- A variable name cannot be a keyword (e.g., if). The set of keywords is finite.

Let $L$ be the set of all valid variable names in our restricted version of Java.

(a) Let $L_0$ be the set of strings satisfying the first 3 conditions above; i.e., we do not require the last condition. How many strings are in $L_0$? You can just give a general formula for your answer; you do not need to give a single number.

**Answer:** Recall that for any set $A$, we let $|A|$ denote the number of elements in $A$. Let $\Sigma_1 = \{\text{a}, \text{b}, \ldots, \text{z}, \text{A}, \text{B}, \ldots, \text{Z}\}$ be the set of upper-case and lower-case Roman letters, and note that $|\Sigma_1| = 52$. Let $\Sigma_2 = \{0, 1, 2, \ldots, 9\}$, which is the set of Arabic numerals, and note that $|\Sigma_2| = 10$. Let $\Sigma_3 = \Sigma_1 \cup \Sigma_2$. Since $\Sigma_1 \cap \Sigma_2 = \emptyset$, $|\Sigma_3| = |\Sigma_1| + |\Sigma_2| = 62$.

Let $L_i$ be all of the strings of length $i$ in $L_0$. Note that $L_1$ consists of all 1-letter strings in $L_0$, so $L_1$ consists of all the single letters in $\Sigma_1$, and $|L_1| = 52$. Also, $L_2$ consists of all 2-letter strings in $L_0$, and if $w \in L_2$, then the first letter of $w$ is from $\Sigma_1$, and the second letter of $w$ is from $\Sigma_3$, so $|L_2| = 52 \times 62$. In general $L_i$ consists of all strings that have first letter from $\Sigma_1$ and the remaining $i - 1$ letters from $\Sigma_3$, so $|L_i| = 52 \times 62^{i-1}$.

Note that

$$L_0 = L_1 \cup L_2 \cup \cdots \cup L_8.$$

Also, $L_i$ and $L_j$ are disjoint for $i \neq j$, so

$$|L_0| = |L_1| + |L_2| + \cdots + |L_8| = \sum_{i=1}^{8} 52 \times 62^{i-1}.$$

(b) Prove that $L$ is finite, where $L$ is the set of strings satisfying all four conditions.

**Answer:** Note that $L \subset L_0$, so we must have that $|L| \leq |L_0|$. In the previous part, we showed that $|L_0| < \infty$, so we must have that $|L| < \infty$.

5. Let $S$ be any set of strings. Prove that $S^* = S^+$ if and only if $\varepsilon \in S$.

**Answer:** Recall

$$S^* = \{\, x_1 x_2 \cdots x_k \mid k \geq 0 \text{ and each } x_i \in S \,\},$$
$$S^+ = \{\, x_1 x_2 \cdots x_k \mid k \geq 1 \text{ and each } x_i \in S \,\},$$

where the concatenation of $k = 0$ strings is $\varepsilon$, so we always have $\varepsilon \in S^*$. Now $S^+$ and $S^*$ are the same except $S^+$ doesn't include the case $k = 0$, so we can write $S^* = S^+ \cup \{\varepsilon\}$. Hence, $S^* = S^+$ if and only if $\varepsilon \in S^+$. Thus, proving that $\varepsilon \in S^+$ if and only if $\varepsilon \in S$ will establish the result in the problem.

Suppose that $\varepsilon \in S$. Then clearly taking $k = 1$ and $x_1 = \varepsilon \in S$ shows that $\varepsilon \in S^+$. Thus, if $\varepsilon \in S$, then $\varepsilon \in S^+$.

Now we need to show the converse: if $\varepsilon \in S^+$, then $\varepsilon \in S$. This is equivalent to its contrapositive: if $\varepsilon \notin S$, then $\varepsilon \notin S^+$. But if $\varepsilon \notin S$, then we cannot concatenate $k \geq 1$ strings $x_1, x_2, \ldots, x_k \in S$, all of which are nonempty since $\varepsilon \notin S$, to obtain the empty string $\varepsilon$. Thus, we have shown $\varepsilon \notin S$ implies $\varepsilon \notin S^+$, so the proof is complete.