Homework 4 Solutions

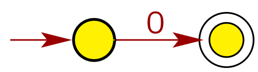
1. Use the procedure described in Lemma 1.55 to convert the regular expression

$$(((00)^*(11)) \cup 01)^*$$

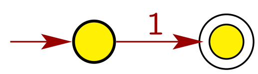
into an NFA.

Answer:

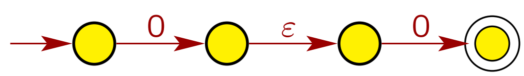
0



1



00



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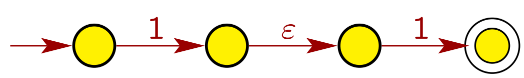
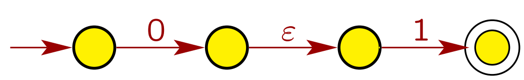
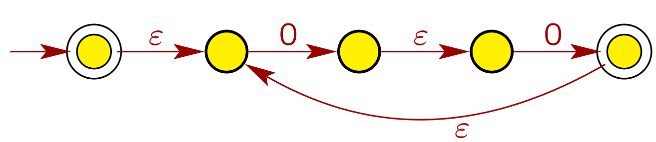


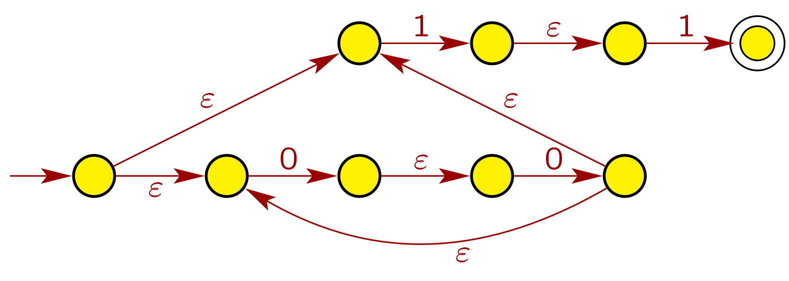
01



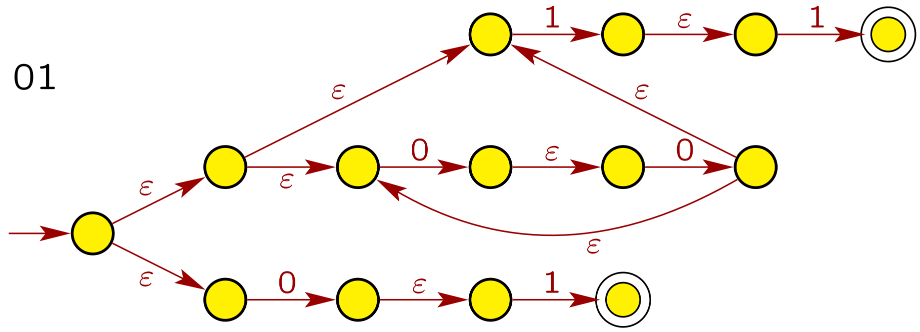
$(00)^*$



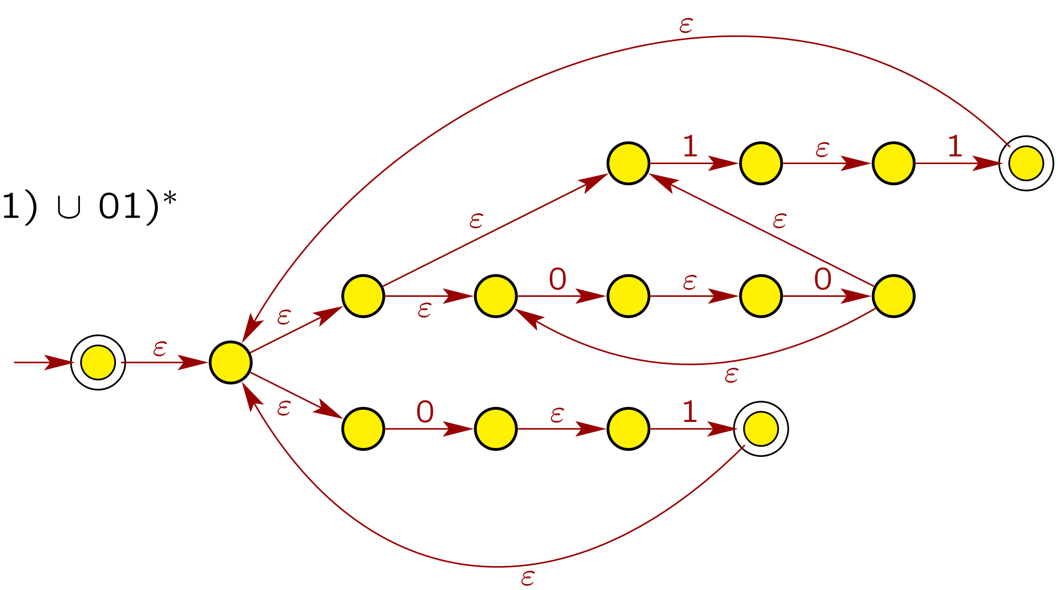
$(00)^*(11)$



$(00)^*(11) \cup 01$

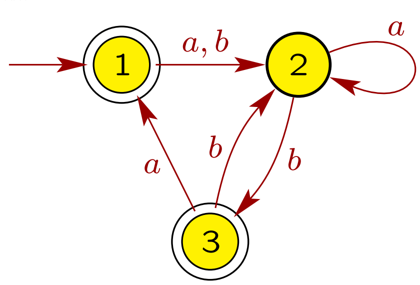


$((00)^*(11) \cup 01)^*$

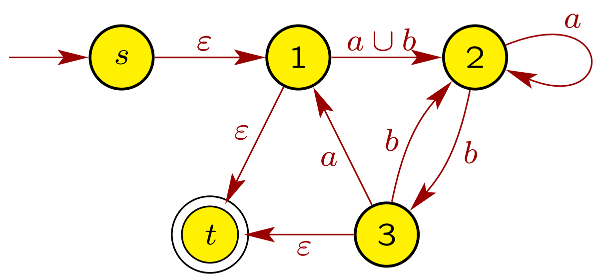


While the resulting NFA is correct, the language has simpler NFAs (with fewer states and transitions).

2. Use the procedure described in Lemma 1.60 to convert the following DFA M to a regular expression.



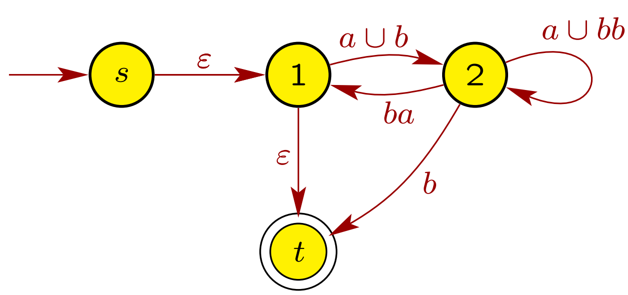
Answer: First convert DFA M into an equivalent GNFA G.



Next, we eliminate the states of G (except for s and t) one at a time. The order in which the states are eliminated does not matter. However, eliminating states in a different order from what is done below may result in a different (but also correct) regular expression. We arbitrarily choose to first eliminate state 3. To do this, we define the set $C = \{2\}$ of states with edges directly into state 3, and set $D = \{1, 2, t\}$ of states with edges directly from state 3. We then need to account for paths that start in a state in C, go immediately to state 3, and then go immediately to a state in D. Thus, we need to account for the paths

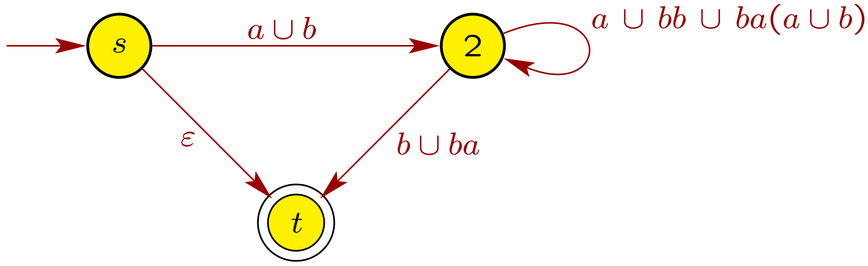
- $2 \to 3 \to 1$, which will create an arc from 2 to 1 labelled with ba;
- $2 \to 3 \to 2$, which will create an arc from 2 to 2 labelled with bb; and
- $2 \to 3 \to t$, which will create an arc from 2 to t labelled with $b\varepsilon = b$.

We combine the previous arc from 2 to 2 labelled a with the new one labelled bb to get the new label $a \cup bb$.

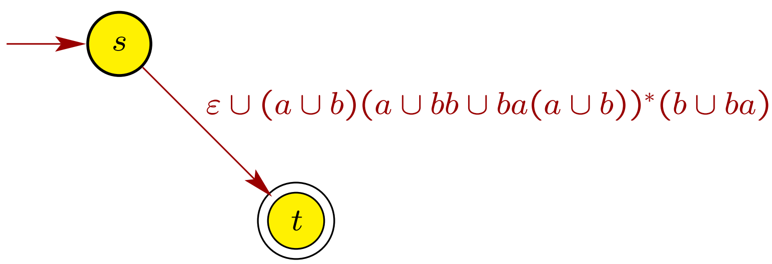


We next eliminate state 1. To do this, we define $C = \{s, 2\}$ as the set of states with edges directly into state 1, and $D = \{2, t\}$ as the set of states with edges directly from state 1. Thus, need to account for the following paths from a state in C, going immediately to state 1, and then immediately going to a state in D:

- $s \to 1 \to 2$, which will create an arc from s to 2 labelled with $\varepsilon(a \cup b) = a \cup b$.
- $s \to 1 \to t$, which will create an arc from s to t labelled with $\varepsilon\varepsilon = \varepsilon$.
- $2 \to 1 \to 2$, which will create an arc from 2 to 2 labelled with $ba(a \cup b)$. We combine this with the existing 2 to 2 arc to get the new label $a \cup bb \cup ba(a \cup b)$.
- $2 \to 1 \to t$, which will create an arc from 2 to t labelled with $ba\varepsilon = ba$. We combine this arc with the existing arc from 2 to t to get the new label $b \cup ba$.



Finally, we eliminate state 2 by adding an arc from s to t labelled $(a \cup b)(a \cup bb \cup ba(a \cup b))^*(b \cup ba)$. We then combine this with the existing s to t arc to get the new label $\varepsilon \cup (a \cup b)(a \cup bb \cup ba(a \cup b))^*(b \cup ba)$.



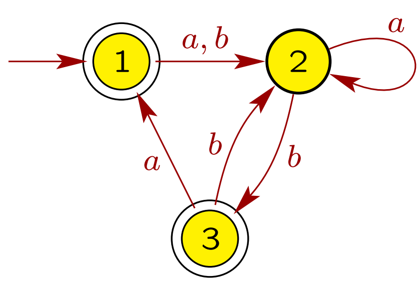
So a regular expression for the language $L(M)$ recognized by the DFA M is

$$\varepsilon \cup (a \cup b)(a \cup bb \cup ba(a \cup b))^*(b \cup ba).$$

Writing this as

$$\underbrace{\varepsilon}_{\text{stay in 1}} \cup \underbrace{(a \cup b)}_{\text{1 to 2}} \underbrace{(a \cup bb \cup ba(a \cup b))^*}_{(\text{2 to 2})^*} \underbrace{(b \cup ba)}_{\text{end in 3 or 1}}$$

should make it clear how the regular expression accounts for every path that starts in 1 and ends in either 3 or 1, which are the accepting states of the given DFA.



3. Each of the following languages is either regular or nonregular. If a language is regular, give a DFA and regular expression for it. If a language is nonregular, give a proof.

   (a) $A_1 = \{\, www \mid w \in \{a, b\}^* \,\}$

Answer: A_1 is nonregular. To prove this, suppose that A_1 is a regular language. Let p be the "pumping length" of the pumping lemma (Theorem 1.70). Consider the string $s = a^p b a^p b a^p b$. Note that $s \in A_1$ since $s = (a^p b)^3$, and $|s| = 3(p+1) \geq p$, so the pumping lemma will hold. Thus, we can split the string s into 3 parts $s = xyz$ satisfying the properties

i. $xy^i z \in A_1$ for each $i \geq 0$,

ii. $|y| > 0$,

iii. $|xy| \leq p$.

Since the first p symbols of s are all a's, the third property implies that x and y consist only of a's. So z will be the rest of the first set of a's, followed by $ba^p ba^p b$. The second property states that $|y| > 0$, so y has at least one a. More precisely, we can then say that

$$\begin{aligned} x &= a^j \text{ for some } j \geq 0, \\ y &= a^k \text{ for some } k \geq 1, \\ z &= a^m b a^p b a^p b \text{ for some } m \geq 0. \end{aligned}$$

Since $a^p b a^p b a^p b = s = xyz = a^j a^k a^m b a^p b a^p b = a^{j+k+m} b a^p b a^p b$, we must have that $j + k + m = p$. The first property implies that $xy^2 z \in A_1$, but

$$\begin{aligned} xy^2 z &= a^j a^k a^k a^m b a^p b a^p b \\ &= a^{p+k} b a^p b a^p b \end{aligned}$$

since $j + k + m = p$. Hence, $xy^2 z \notin A_1$ because $k \geq 1$, and we get a contradiction. Therefore, A_1 is a nonregular language.

(b) $A_2 = \{ w \in \{a, b\}^* \mid w = w^{\mathcal{R}} \}$.

Answer: A_2 is nonregular. To prove this, suppose that A_2 is a regular language. Let p be the "pumping length" of the pumping lemma (Theorem 1.70). Consider the string $s = a^p b a^p$. Note that $s \in A_2$ since $s = s^{\mathcal{R}}$, and $|s| = 2p + 1 \geq p$, so the pumping lemma will hold. Thus, we can split the string s into 3 parts $s = xyz$ satisfying the properties

i. $xy^i z \in A_2$ for each $i \geq 0$,

ii. $|y| > 0$,

iii. $|xy| \leq p$.

Since the first p symbols of s are all a's, the third property implies that x and y consist only of a's. So z will be the rest of the first set of a's, followed by ba^p. The second property states that $|y| > 0$, so y has at least one a. More precisely, we can then say that

$$\begin{aligned} x &= a^j \text{ for some } j \geq 0, \\ y &= a^k \text{ for some } k \geq 1, \\ z &= a^m b a^p \text{ for some } m \geq 0. \end{aligned}$$

Since $a^pba^p = s = xyz = a^ja^ka^mba^p = a^{j+k+m}ba^p$, we must have that $j + k + m = p$. The first property implies that $xy^2z \in A_2$, but

$$\begin{aligned} xy^2z &= a^ja^ka^ka^mba^p \\ &= a^{p+k}ba^p \end{aligned}$$

since $j + k + m = p$. Hence, $xy^2z \notin A_2$ because $(a^{p+k}ba^p)^{\mathcal{R}} = a^pba^{p+k} \neq a^{p+k}ba^p$ since $k \geq 1$, and we get a contradiction. Therefore, A_2 is a nonregular language.

(c) $A_3 = \{\, a^{2n}b^{3n}a^n \mid n \geq 0 \,\}$.

Answer: A_3 is nonregular. To prove this, suppose that A_3 is a regular language. Let p be the "pumping length" of the pumping lemma (Theorem 1.70). Consider the string $s = a^{2p}b^{3p}a^p$. Note that $s \in A_3$, and $|s| = 6p \geq p$, so the pumping lemma will hold. Thus, we can split the string s into 3 parts $s = xyz$ satisfying the properties

i. $xy^iz \in A_3$ for each $i \geq 0$,

ii. $|y| > 0$,

iii. $|xy| \leq p$.

Since the first p symbols of s are all a's, the third property implies that x and y consist only of a's. So z will be the rest of the first set of a's, followed by $b^{3p}a^p$. The second property states that $|y| > 0$, so y has at least one a. More precisely, we can then say that

$$\begin{aligned} x &= a^j \text{ for some } j \geq 0, \\ y &= a^k \text{ for some } k \geq 1, \\ z &= a^{m+p}b^{3p}a^p \text{ for some } m \geq 0. \end{aligned}$$

Since $a^{2p}b^{3p}a^p = s = xyz = a^ja^ka^{m+p}b^{3p}a^p = a^{j+k+m+p}b^{3p}a^p$, we must have that $j + k + m + p = 2p$, or equivalently, $j + k + m = p$, so $j + k \leq p$. The first property implies that $xy^2z \in A_3$, but

$$\begin{aligned} xy^2z &= a^ja^ka^ka^{m+p}b^{3p}a^p \\ &= a^{2p+k}b^{3p}a^p \end{aligned}$$

since $j + k + m = p$. Hence, $xy^2z \notin A_3$ because $k \geq 1$, and we get a contradiction. Therefore, A_3 is a nonregular language.

(d) $A_4 = \{\, w \in \{a, b\}^* \mid w \text{ has more } a\text{'s than } b\text{'s} \,\}$.

Answer: A_3 is nonregular. To prove this, suppose that A_4 is a regular language. Let p be the "pumping length" of the pumping lemma (Theorem 1.70). Consider the string $s = b^pa^{p+1}$. Note that $s \in A_4$, and $|s| = 2p+1 \geq p$, so the pumping lemma will hold. Thus, we can split the string s into 3 parts $s = xyz$ satisfying the properties

i. $xy^iz \in A_4$ for each $i \geq 0$,

ii. $|y| > 0$,

iii. $|xy| \leq p$.

Since the first p symbols of s are all b's, the third property implies that x and y consist only of b's. So z will be the rest of the b's, followed by a^{p+1}. The second property states that $|y| > 0$, so y has at least one b. More precisely, we can then say that

$$\begin{aligned} x &= b^j \text{ for some } j \geq 0, \\ y &= b^k \text{ for some } k \geq 1, \\ z &= b^m a^{p+1} \text{ for some } m \geq 0. \end{aligned}$$

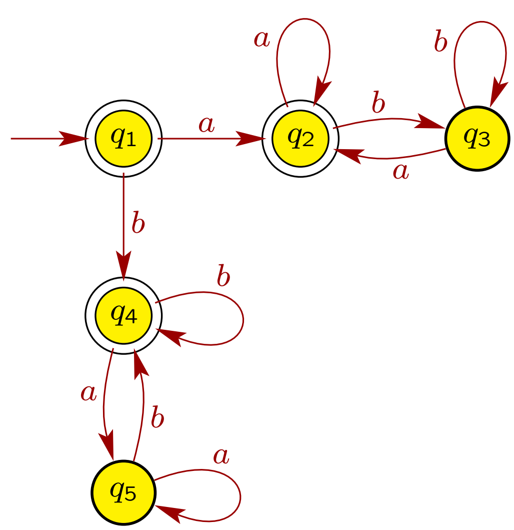
Since $b^p a^{p+1} = s = xyz = b^j b^k b^m a^{p+1} = b^{j+k+m} a^{p+1}$, we must have that $j + k + m = p$. The first property implies that $xy^2z \in A_4$, but

$$\begin{aligned} xy^2z &= b^j b^k b^k b^m a^{p+1} \\ &= b^{p+k} a^{p+1} \end{aligned}$$

since $j + k + m = p$. Hence, $xy^2z \notin A_4$ because it doesn't have more a's than b's since $k \geq 1$, and we get a contradiction. Therefore, A_4 is a nonregular language.

(e) $A_5 = \{\, w \in \{a, b\}^* \mid n_{ab}(w) = n_{ba}(w) \,\}$, where $n_s(w)$ is the number of occurrences of the substring $s \in \{a, b\}^*$ in w.

Answer: A_5 is regular. A regular expression for the language is $a(a \cup bb^*a)^* \cup b(b \cup aa^*b)^* \cup \varepsilon$. Another regular expression is $a(a \cup b)^*a \cup b(a \cup b)^*b \cup a \cup b \cup \varepsilon$. A DFA for the language is



There are infinitely many other correct regular expressions and DFAs for A_5.

(f) $A_6 = \{ w \in \{a, b\}^* \mid n_a(w) \neq n_b(w) \}$.

Answer: A_6 is nonregular, which we will establish via a proof by contradiction. Suppose that A_6 were regular. Then its complement $\overline{A_6}$ is regular because the class of regular languages is closed under complementation (HW 2, problem 3), where

$$\overline{A_6} = \{ w \in \{a, b\}^* \mid n_a(w) = n_b(w) \}. \tag{1}$$

By arguing as on slide 1-112 of the lecture notes, we can show that $\overline{A_6}$ is nonregular (which we could also prove by applying the pumping lemma with the string $s = a^p b^p \in \overline{A_6}$), giving a contradiction, so A_6 is nonregular.

Alternatively, using an approach outlined at this link, we can also prove that A_6 is nonregular via the pumping lemma (Theorem 1.70) by carefully constructing a string $s \in A_6$ to get a contradiction. For a contradiction, suppose that A_6 is regular, and let p be the pumping length of the pumping lemma. Consider the string $s = a^p b^{p!+p}$, where $p! = p(p-1)(p-2)\cdots 1$ is p factorial, so $s \in A_6$ because $n_a(s) = p \neq p! + p = n_b(s)$. Also, $|s| = 2p + p! \geq p$, so the conclusions of the pumping lemma will hold. Thus, we can split the string s into 3 parts $s = xyz$ satisfying the properties

i. $xy^iz \in A_6$ for each $i \geq 0$,

ii. $|y| > 0$,

iii. $|xy| \leq p$.

Since the first p symbols of s are all a's, the third property implies that x and y consist only of a's. So z will be the rest of the a's, followed by $b^{p!+p}$. The second property states that $|y| > 0$, so y has at least one a. More precisely, we can then say that

$$\begin{aligned} x &= a^j \text{ for some } j \geq 0, \\ y &= a^k \text{ for some } k \geq 1, \\ z &= a^m b^{p!+p} \text{ for some } m \geq 0, \end{aligned}$$

with $j + k \leq p$, so

$$1 \leq k \leq p. \tag{2}$$

Also, we must have that

$$j + k + m = p \tag{3}$$

because $a^p b^{p!+p} = s = xyz = a^j a^k a^m b^{p!+p} = a^{j+k+m} b^{p!+p}$.

The first property of the pumping lemma states that for all $i \geq 0$, the pumped string xy^iz must be in A_6, and we will get a contradiction by showing that no matter the value of k satisfying equation (2), there will exist an $i \geq 0$ (depending on k) so that the pumped string $xy^iz \notin A_6$, which means that $n_a(xy^iz) =$

$n_b(xy^iz)$ for this value of i by equation (1). Now this pumped string is $xy^iz = a^j a^{ik} a^m b^{p!+p}$, with $n_a(xy^iz) = j + ik + m$ and $n_b(xy^iz) = p! + p$. To get our contradiction, we must show that for each k satisfying equation (2), there exists an $i \geq 0$ such that

$$j + ik + m = p! + p \tag{4}$$

so that the pumped string $xy^iz \notin A_6$ for this i because $n_a(xy^iz) = n_b(xy^iz)$. Now equation (3) implies that the left side of equation (4) satisfies $j + ik + m = j + k + m + (i-1)k = p + (i-1)k$, so equation (4) is equivalent to $p + (i-1)k = p! + p$, which is the same as $(i-1)k = p!$. Thus, for any k satisfying equation (2), taking $i = 1 + (p!/k)$, which is an integer because $k \in \{1, 2, \ldots, p\}$, results in the pumped string $xy^iz \notin A_6$, contradicting the pumping lemma's first property. Hence, A_6 is nonregular.

(g) $A_7 = \{ a^j b^k c^l \mid j, k, l \geq 0 \text{ and if } j = 1, \text{ then } k = l \}$.

Answer: A_7 is nonregular, which we will establish via a proof by contradiction. Suppose that A_7 is regular, and consider the language L having regular expression ab^*c^*, so L is regular by Kleene's Theorem. Let $D = A_7 \cap L$, which must then be regular if A_7 is regular because the class of regular languages is closed under intersection (HW 2, problem 5). Now

$$D = A_7 \cap L = \{ ab^n c^n \mid n \geq 0 \},$$

which we will show below is nonregular, giving a contradiction, so A_7 must be nonregular.

To show that D is nonregular, we will use the pumping lemma (Theorem 1.70). For a contradiction, suppose that D is regular, and let p be the pumping length of the pumping lemma. Consider the string $s = ab^pc^p \in D$, where $|s| = 2p + 1 \geq p$, so the conclusions of the pumping lemma will hold. Thus, we can split the string s into 3 parts $s = xyz$ satisfying the properties

i. $xy^iz \in A_7$ for each $i \geq 0$,

ii. $|y| > 0$,

iii. $|xy| \leq p$.

The second property implies that $y \neq \varepsilon$. Also, since the first p symbols of s are ab^{p-1}, the third property implies that y is a nonempty substring of ab^{p-1}. We can break down all of the nonempty possibilities for y into two cases: y contains a, or y does not contain a. We next show a contradiction arises in each case.

- Suppose y contains a. The first conclusion of the pumping lemma implies that $xy^0z = xz \in D$, but since y has the only a in s, the pumped-down string xz does not begin with a, so $xz \notin D$, which is a contradiction.
- Suppose that y does not contain a. Since $y \neq \varepsilon$ and y is a nonempty substring of ab^{p-1}, we must have that y has at least one b. The first property of the pumping lemma implies that $xy^0z = xz \in D$, but since y has at least one b, the pumped-down string xz has fewer b's than c's, so $xz \notin D$, which is a contradiction.

The two above cases cover all possibilities for y satisfying the three properties of the pumping lemma, and each gives a contradiction. Thus, the pumping lemma does not hold, so D is nonregular.

An interesting property about A_7 is that while it is nonregular, this cannot be proven using the pumping lemma (Theorem 1.70). In particular, there exists some p (specifically, we can take $p = 3$) such that for every string $s \in A_7$ with $|s| \geq p$, we can split $s = xyz$ such that (i) $xy^iz \in A_7$ for all $i \geq 0$, (ii) $y \neq \varepsilon$, and (iii) $|xy| \leq p$. To see why, write $A_7 = E \cup F$ with

$$E = \{ ab^nc^n \mid n \geq 0 \},$$
$$F = \{ a^jb^kc^l \mid j, k, l \geq 0 \text{ and } j \neq 1 \},$$

so $E \subseteq A_7$ and $F \subseteq A_7$. We can cover all possibilities for strings $s \in A_7$ with $|s| \geq p$ through two cases: either $s \in E$ or $s \in F$ (or both)

- Suppose that $s \in E$ with $|s| \geq p = 3$, so $s = ab^nc^n$ for some $n \geq 1$. We then split s as $s = xyz$, with $x = \varepsilon$, $y = a$, and $z = b^nc^n$. Note that $|xy| = |a| = 1 \leq p = 3$ and $y \neq \varepsilon$, so the second and third properties of the pumping lemma hold. For each $i \geq 0$, the pumped string xy^iz is $a^ib^nc^n$, which is in $F \subseteq A_7$ for $i \neq 1$, and belongs to $E \subseteq A_7$ for $i = 1$, so the pumped string $xy^iz \in A_7$ for all $i \geq 0$. In other words, each string $s \in E$ with $|s| \geq p = 3$ can be split into $s = xyz$ with all three conclusions of the pumping lemma holding.
- Suppose that $s \in F$ with $|s| \geq p = 3$, so $s = a^jb^kc^l$ for some $j, k, l \geq 0$ with $j \neq 1$. Then all possibilities for s belong to one of the following three subcases: $j = 0$, $j = 2$, and $j \geq 3$.
  - If $j = 0$, then $s = b^kc^l$, with $|s| \geq p$, so $s = s_1s_2 \cdots s_n$ for some $n \geq p$ with each $s_i \in \{b, c\}$, and all b's appear before any c. Split s into $s = xyz$ with $x = \varepsilon$, $y = s_1$, and $z = s_2s_3 \cdots s_n$. Thus, $y \neq \varepsilon$ and $|xy| = |s_1| = 1 \leq p = 3$, so the second and third properties of the pumping lemma hold. For each $i \geq 0$, the pumped string is $xy^iz = s_1^i s_2 \cdots s_n$, which can be generated by the regular expression b^*c^*, so the pumped string will always be in $F \subseteq A_7$.
  - If $j = 2$, then the string is $s = aab^kc^l$. We then can split s into $s = xyz$ with $x = \varepsilon$, $y = aa$, and $z = b^kc^l$. Thus, $y \neq \varepsilon$ and $|xy| = |aa| = 2 \leq p = 3$, so the second and third properties of the pumping lemma hold. Now for each $i \geq 0$, the pumped string $xy^iz = (aa)^ib^kc^l$ has $2i$ a's at the beginning followed by b^kc^l, so the number of a's is never 1, implying that $xy^iz \in F \subseteq A_7$.
  - If $j \geq 3$, then the string $s = a^jb^kc^l$ has at least 3 a's at the beginning. We then can split s into $s = xyz$ with $x = \varepsilon$, $y = a$, and $z = a^{j-1}b^kc^l$. Thus, $y \neq \varepsilon$ and $|xy| = |a| = 1 \leq p = 3$, so the second and third properties of the pumping lemma hold. Now for each $i \geq 0$, the pumped string $xy^iz = a^{i+j-1}b^kc^l$ has $i + j - 1 \geq 2$ a's at the beginning followed by b^kc^l, implying that $xy^iz \in F \subseteq A_7$.

Thus, both cases of $s \in E$ and $s \in F$ always can be pumped to get strings still in A_7, so there is no contradiction. In other words, A_7 is nonregular, but we cannot prove this via the pumping lemma.

To understand this, recall that if the conclusions of the pumping lemma do not hold for a given language, then the language is nonregular. While this gives a *sufficient* condition to ensure a language is nonregular, language A_7 shows that the condition is not *necessary* for nonregularity. (A *necessary and sufficient* condition for regularity is provided by the Myhill-Nerode theorem, which our class doesn't cover but appears as problem 1.52 of the Sipser book.)

4. Suppose that language A is recognized by an NFA N, and language B is the collection of strings *not* accepted by some DFA M. Prove that $A \circ B$ is a regular language.

   Answer: Since A is recognized by an NFA, we know that A is regular since a language is regular if and only if it is recognized by an NFA (Corollary 1.20). Note that the DFA M recognizes the language $\overline{B}$, the complement of B. Since $\overline{B}$ is recognized by a DFA, by definition, $\overline{B}$ is regular. We know from a problem on the previous homework that $\overline{B}$ being regular implies that its complement $\overline{\overline{B}}$ is regular. ($\overline{\overline{B}}$ is the complement of the complement of B.) But $\overline{\overline{B}} = B$, so B is regular. Since A and B are regular, their concatenation $A \circ B$ is regular by Theorem 1.23.

5. (a) Prove that if we add a finite set of strings to a regular language, the result is a regular language.

   Answer: Let A be a regular language, and let B be a finite set of strings. We know from class (see page 1-95 of Lecture Notes for Chapter 1) that finite languages are regular, so B is regular. Thus, $A \cup B$ is regular since the class of regular languages is closed under union (Theorem 1.22).

   (b) Prove that if we remove a finite set of strings from a regular language, the result is a regular language.

   Answer: Let A be a regular language, and let B be a finite set of strings with $B \subseteq A$. Let C be the language resulting from removing B from A, i.e., $C = A - B$. As we argued in the previous part, B is regular. Note that $C = A - B = A \cap \overline{B}$. Since B is regular, $\overline{B}$ is regular since the class of regular languages is closed under complement. We proved in an earlier homework that the class of regular languages is closed under intersection, so $A \cap \overline{B}$ is regular since A and $\overline{B}$ are regular. Therefore, $A - B$ is regular.

   (c) Prove that if we add a finite set of strings to a nonregular language, the result is a nonregular language.

   Answer: Let A be a nonregular language, and let B be a finite set of strings. We want to add B to A, so we may assume that none of the strings in B are in A, i.e., $A \cap B = \emptyset$. Let C be the language obtained by adding B to A, i.e.,

$C = A \cup B$. Suppose for a contradiction that C is regular, and we now show this is impossible. Since $A \cap B = \emptyset$, we have that $A = C - B$. Since C and B are regular (the latter because B is finite), the previous part of this problem implies that $C - B = C \cap \overline{B}$ must be regular, but we assumed that $A = C - B$ is nonregular, so we get a contradiction.

(d) Prove that if we remove a finite set of strings from a nonregular language, the result is a nonregular language.

Answer: Let A be a nonregular language, and let B be a finite set of strings, where $B \subseteq A$. Let C be the language obtained by removing B from A, i.e., $C = A - B$. Suppose that C is regular, and we now show this is impossible. Since we removed B from A to get C, we must have that $C \cap B = \emptyset$, so $A = C \cup B$. Now C is regular by assumption and B is regular since it's finite, so $C \cup B$ must be regular by Theorem 1.25. But we assumed that $A = C \cup B$ is nonregular, so we get a contradiction.

6. Consider the following statement: "If A is a nonregular language and B is a language such that $B \subseteq A$, then B must be nonregular." If the statement is true, give a proof. If it is not true, give a counterexample showing that the statement doesn't always hold.

Answer: The statement is not always true. For example, we know that the language $A = \{ \mathtt{0}^j \mathtt{1}^j \mid j \geq 0 \}$ is nonregular. Define the language $B = \{\mathtt{01}\}$, and note that $B \subseteq A$. However, B is finite, so we know that it is regular.