CS 341: Foundations of Computer Science II
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Homework 1 Solutions

1. We are given the sets of strings:

$$\begin{aligned} C &= \{\varepsilon, aab, baa\}, \\ D &= \{bb, aab\}, \\ E &= \{\varepsilon\}, \\ F &= \emptyset. \end{aligned}$$

   (a) $D \cup C = \{\varepsilon, bb, aab, baa\}$

   (b) $C \cup F = \{\varepsilon, aab, baa\} = C$

   (c) $C \times D = \{\ (\varepsilon, bb),\ (\varepsilon, aab),\ (aab, bb),\ (aab, aab),\ (baa, bb),\ (baa, aab)\ \}$

   (d) $C \cap D = \{aab\}$

   (e) $D \circ C = \{bb, aab, bbaab, bbbaa, aabaab, aabbaa\}$

   (f) $C \circ E = \{\varepsilon, aab, baa\}$

   (g) $D \circ D \circ D = \{bbbbbb, aabbbbb, bbaabbb, bbbbaab, aabaabbb, aabbbaab, bbaabaab, aabaabaab\}$

   (h) $\mathcal{P}(C) = \{\ \emptyset,\ \{\varepsilon\},\ \{aab\},\ \{baa\},\ \{\varepsilon, aab\},\ \{\varepsilon, baa\},\ \{aab, baa\},\ \{\varepsilon, aab, baa\}\ \}$

   (i) $D - C = \{bb\}$

   (j) $C^+ = \{\varepsilon, aab, baa, aabaab, aabbaa, baaaab, baabaa, aabaabaab, \ldots\}$

   (k) $F^* = \{\varepsilon\}$

   (l) $E \subseteq C$ since every element of E is also in C.

   (m) $D \not\subseteq C$ since $bb \in D$ but $bb \notin C$.

   (n)
   - C is closed under reversal since $\varepsilon^{\mathcal{R}} = \varepsilon \in C$, $(aab)^{\mathcal{R}} = baa \in C$, and $(baa)^{\mathcal{R}} = aab \in C$.
   - D is not closed under reversal since $(aab)^{\mathcal{R}} = baa \notin D$.
   - E is closed under reversal since $\varepsilon^{\mathcal{R}} = \varepsilon \in E$.

2. (a) It is not true in general that $w \in S$. For example, suppose that $w = aa$, $S = \{a\}$, and $T = \{a, aa\}$. Then note that $T = S \cup \{aa\}$, and $S^* = T^* = \{\varepsilon, a, aa, aaa, \ldots\}$, but $aa \notin S$.

   (b) It must be the case that $w \in S^*$. Note that $w \in T$, so $w \in T^*$ since any string in T is also in T^* because $T \subset T^*$. But since $T^* = S^*$, we must have that $w \in S^*$.

3. (a) Let $S = \{\varepsilon, a\}$. Then $S^* = \{\varepsilon, a, aa, aaa, \ldots\}$ and $S^+ = \{\varepsilon, a, aa, aaa, \ldots\}$, so $S^* = S^+$.
   (b) Let $S = \{a\}$. Then $S^* = \{\varepsilon, a, aa, aaa, \ldots\}$ and $S^+ = \{a, aa, aaa, \ldots\}$, so $S^* \neq S^+$.
   (c) Let $S = \{\varepsilon, a, aa, aaa, \ldots\}$. Then $S^* = \{\varepsilon, a, aa, aaa, \ldots\}$, so $S = S^*$.
   (d) Let $S = \{a\}$. Then $S^* = \{\varepsilon, a, aa, aaa, \ldots\}$, so $S \neq S^*$.
   (e) Let $S = \{\varepsilon\}$. Then $S^* = \{\varepsilon\}$, so S^* is finite.

4. (a) Recall that for any set A, we let $|A|$ denote the number of elements in A. Let $\Sigma_1 = \{\mathrm{a}, \mathrm{b}, \ldots, \mathrm{z}, \mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}\}$ be the set of upper-case and lower-case Roman letters, and note that $|\Sigma_1| = 52$. Let $\Sigma_2 = \{0, 1, 2, \ldots, 9\}$, which is the set of Arabic numerals, and note that $|\Sigma_2| = 10$. Let $\Sigma_3 = \Sigma_1 \cup \Sigma_2$. Since $\Sigma_1 \cap \Sigma_2 = \emptyset$, $|\Sigma_3| = |\Sigma_1| + |\Sigma_2| = 62$.
   Let L_i be all of the strings of length i in L_0. Note that L_1 consists of all 1-letter strings in L_0, so L_1 consists of all the single letters in Σ_1, and $|L_1| = 52$. Also, L_2 consists of all 2-letter strings in L_0, and if $w \in L_2$, then the first letter of w is from Σ_1, and the second letter of w is from Σ_3, so $|L_2| = 52 \times 62$. In general L_i consists of all strings that have first letter from Σ_1 and the remaining $i-1$ letters from Σ_3, so $|L_i| = 52 \times 62^{i-1}$.
   Note that
   $$L_0 = L_1 \cup L_2 \cup \cdots \cup L_8.$$
   Also, L_i and L_j are disjoint for $i \neq j$, so
   $$|L_0| = |L_1| + |L_2| + \cdots + |L_8| = \sum_{i=1}^{8} 52 \times 62^{i-1}.$$
   (b) Note that $L \subset L_0$, so we must have that $|L| \leq |L_0|$. In the previous part, we showed that $|L_0| < \infty$, so we must have that $|L| < \infty$.

5. Recall
$$S^* = \{\, x_1 x_2 \cdots x_k \mid k \geq 0 \text{ and each } x_i \in S \,\},$$
$$S^+ = \{\, x_1 x_2 \cdots x_k \mid k \geq 1 \text{ and each } x_i \in S \,\},$$
where the concatenation of $k = 0$ strings is ε, so we always have $\varepsilon \in S^*$. Now S^+ and S^* are the same except S^+ doesn't include the case $k = 0$, so we can write $S^* = S^+ \cup \{\varepsilon\}$. Hence, $S^* = S^+$ if and only if $\varepsilon \in S^+$. Thus, proving that $\varepsilon \in S^+$ if and only if $\varepsilon \in S$ will establish the result in the problem.

Suppose that $\varepsilon \in S$. Then clearly taking $k = 1$ and $x_1 = \varepsilon \in S$ shows that $\varepsilon \in S^+$. Thus, if $\varepsilon \in S$, then $\varepsilon \in S^+$.

Now we need to show the converse: if $\varepsilon \in S^+$, then $\varepsilon \in S$. This is equivalent to its contrapositive: if $\varepsilon \notin S$, then $\varepsilon \notin S^+$. But if $\varepsilon \notin S$, then we cannot concatenate $k \geq 1$ strings $x_1, x_2, \ldots, x_k \in S$, all of which are nonempty since $\varepsilon \notin S$, to obtain the empty string ε. Thus, we have shown $\varepsilon \notin S$ implies $\varepsilon \notin S^+$, so the proof is complete.