1. For the drawing below,

we formally express the DFA as $M = (Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, q_3\}$
- $\Sigma = \{a, b\}$
- transition function $\delta$ is given by

$$
\begin{array}{c|cc}
   & a & b \\
\hline
q_1 & q_1 & q_2 \\
q_2 & q_1 & q_3 \\
q_3 & q_1 & q_3 \\
\end{array}
$$

- $q_1$ is the start state
- $F = \{q_1, q_3\}$ is the set of accept states.

2. There are (infinitely) many correct DFAs for each part below.

(a) A DFA that recognizes the language $A = \{\varepsilon, b, ab\}$ is
We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, \ldots, q_8\}$
- $\Sigma = \{a, b\}$
- transition function $\delta$ is given by
- $q_1$ is the start state
- $F = \{q_1, q_3, q_5\}$ is the set of accept states.

There are simpler DFAs that recognize this language. Can you come up with one with only 4 states?

(b) A DFA that recognizes the language $B = \{ w \in \Sigma^* \mid n_a(w) \mod 3 = 1 \}$ is
We formally express the DFA as a 5-tuple \((Q, \Sigma, \delta, q_1, F)\), where

- \(Q = \{q_1, q_2, q_3\}\)
- \(\Sigma = \{a, b\}\)
- transition function \(\delta\) is given by

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- \(q_1\) is the start state
- \(F = \{q_2\}\) is the set of accept states.

(c) A DFA that recognizes the language \(C = \{ w \in \Sigma^* | w = saba \text{ for some string } s \in \Sigma^* \}\) is

We formally express the DFA as a 5-tuple \((Q, \Sigma, \delta, q_1, F)\), where

- \(Q = \{q_1, q_2, q_3, q_4\}\)
- \(\Sigma = \{a, b\}\)
• transition function $\delta$ is given by

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• $q_1$ is the start state
• $F = \{q_4\}$ is the set of accept states.

(d) Because $D = \overline{C}$, the complement of $C$, we can convert the DFA for $C$ into a DFA for $D$ by swapping the accept and non-accept states:

We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

• $Q = \{q_1, q_2, q_3, q_4\}$
• $\Sigma = \{a, b\}$
• transition function $\delta$ is given by

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• $q_1$ is the start state
• $F = \{q_1, q_2, q_3\}$ is the set of accept states.

(e) A DFA for $E = \{w \in \Sigma^* \mid w$ begins with $b$ and ends with $a\}$ is
We formally express the DFA as a 5-tuple \((Q, \Sigma, \delta, q_1, F)\), where

- \(Q = \{q_1, q_2, q_3, q_4\}\)
- \(\Sigma = \{a, b\}\)
- transition function \(\delta\) is given by

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- \(q_1\) is the start state
- \(F = \{q_3\}\) is the set of accept states.

(f) A DFA for \(F = \{ w \in \Sigma^* \mid n_a(w) \geq 2, n_b(w) \leq 1 \}\) is

We formally express the DFA as a 5-tuple \((Q, \Sigma, \delta, q_1, F)\), where

- \(Q = \{q_1, q_2, \ldots, q_7\}\)
• $\Sigma = \{a, b\}$
• transition function $\delta$ is given by

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• $q_1$ is the start state
• $F = \{q_3, q_6\}$ is the set of accept states.

(g) A DFA for $G = \{ w \in \Sigma^* | |w| \geq 2, \text{second-to-last symbol of } w \text{ is } b \}$ is

We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where
• $Q = \{q_1, q_2, q_3, q_4\}$
• $\Sigma = \{a, b\}$
• transition function $\delta$ is given by

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• $q_1$ is the start state
• $F = \{q_3, q_4\}$ is the set of accept states.

3. Show that, if $M$ is a DFA that recognizes language $B$, swapping the accept and non-accept states in $M$ yields a new DFA that recognizes $\overline{B}$, the complement of $B$. Conclude that the class of regular languages is closed under complement.
Answer:

Suppose language \( B \) over alphabet \( \Sigma \) has a DFA
\[
M = (Q, \Sigma, \delta, q_1, F).
\]

Then, a DFA for the complementary language \( \overline{B} \) is
\[
\overline{M} = (Q, \Sigma, \delta, q_1, Q - F).
\]

The reason why \( \overline{M} \) recognizes \( \overline{B} \) is as follows. First note that \( M \) and \( \overline{M} \) have the same transition function \( \delta \). Thus, since \( M \) is deterministic, \( \overline{M} \) is also deterministic. Now consider any string \( w \in \Sigma^* \). Running \( M \) on input string \( w \) will result in \( M \) ending in some state \( r \in Q \). Since \( M \) is deterministic, there is only one possible state that \( M \) can end in on input \( w \). If we run \( \overline{M} \) on the same input \( w \), then \( \overline{M} \) will end in the same state \( r \) since \( M \) and \( \overline{M} \) have the same transition function. Also, since \( M \) is deterministic, there is only one possible ending state that \( M \) can be in on input \( w \).

Now suppose that \( w \in B \). Then \( M \) will accept \( w \), which means that the ending state \( r \in F \), i.e., \( r \) is an accept state of \( M \). But then \( r \notin Q - F \), so \( \overline{M} \) does not accept \( w \) since \( \overline{M} \) has \( Q - F \) as its set of accept states. Similarly, suppose that \( w \notin B \). Then \( M \) will not accept \( w \), which means that the ending state \( r \notin F \). But then \( r \in Q - F \), so \( \overline{M} \) accepts \( w \). Therefore, \( \overline{M} \) accepts string \( w \) if and only if \( M \) does not accept string \( w \), so \( \overline{M} \) recognizes language \( \overline{B} \). Hence, the class of regular languages is closed under complement.

4. We say that a DFA \( M \) for a language \( A \) is minimal if there does not exist another DFA \( M' \) for \( A \) such that \( M' \) has strictly fewer states than \( M \). Suppose that \( M = (Q, \Sigma, \delta, q_0, F) \) is a minimal DFA for \( A \). Using \( M \), we construct a DFA \( \overline{M} \) for the complement \( \overline{A} \) as \( \overline{M} = (Q, \Sigma, \delta, q_0, Q - F) \). Prove that \( \overline{M} \) is a minimal DFA for \( \overline{A} \).

Answer:
We prove this by contradiction. Suppose that \( \overline{M} \) is not a minimal DFA for \( \overline{A} \). Then there exists another DFA \( D \) for \( \overline{A} \) such that \( D \) has strictly fewer states than \( \overline{M} \). Now create another DFA \( D' \) by swapping the accepting and non-accepting states of \( D \). Then \( D' \) recognizes the complement of \( \overline{A} \). But the complement of \( \overline{A} \) is just \( A \), so \( D' \) recognizes \( A \). Note that \( D' \) has the same number of states as \( D \), and \( \overline{M} \) has the same number of states as \( M \). Thus, since we assumed that \( D \) has strictly fewer states than \( \overline{M} \), then \( D' \) has strictly fewer states than \( M \). But since \( D' \) recognizes \( A \), this contradicts our assumption that \( M \) is a minimal DFA for \( A \). Therefore, \( \overline{M} \) is a minimal DFA for \( \overline{A} \).

5. Suppose \( A_1 \) and \( A_2 \) are defined over the same alphabet \( \Sigma \). Suppose DFA \( M_1 \) recognizes \( A_1 \), where \( M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \). Suppose DFA \( M_2 \) recognizes \( A_2 \), where \( M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \). Define DFA \( M_3 = (Q_3, \Sigma, \delta_3, q_3, F_3) \) for \( A_1 \cap A_2 \) as follows:

- Set of states of \( M_3 \) is
  \[
  Q_3 = Q_1 \times Q_2 = \{ (x, y) \mid x \in Q_1, y \in Q_2 \}.
  \]
The alphabet of $M_3$ is $\Sigma$.

$M_3$ has transition function $\delta_3 : Q_3 \times \Sigma \rightarrow Q_3$ such that for $x \in Q_1$, $y \in Q_2$, and $\ell \in \Sigma$,

$$\delta_3((x, y), \ell) = (\delta_1(x, \ell), \delta_2(y, \ell)).$$

The initial state of $M_3$ is $s_3 = (q_1, q_2) \in Q_3$.

The set of accept states of $M_3$ is

$$F_3 = \{ (x, y) \in Q_1 \times Q_2 \mid x \in F_1 \text{ and } y \in F_2 \} = F_1 \times F_2.$$

Since $Q_3 = Q_1 \times Q_2$, the number of states in the new DFA $M_3$ is $|Q_3| = |Q_1| \cdot |Q_2|$. Thus, $|Q_3| < \infty$ since $|Q_1| < \infty$ and $|Q_2| < \infty$. 
