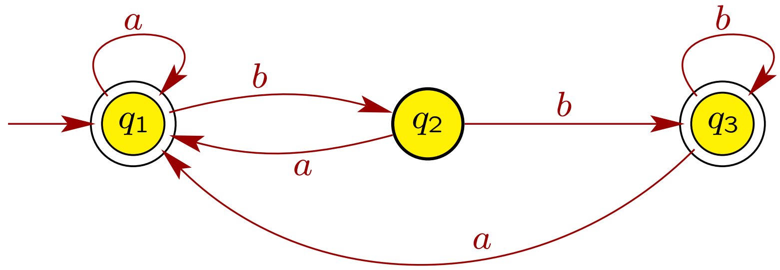
Homework 2 Solutions

1. For the state diagram below,



we formally express the DFA as $M = (Q, \Sigma, \delta, q_1, F)$, where

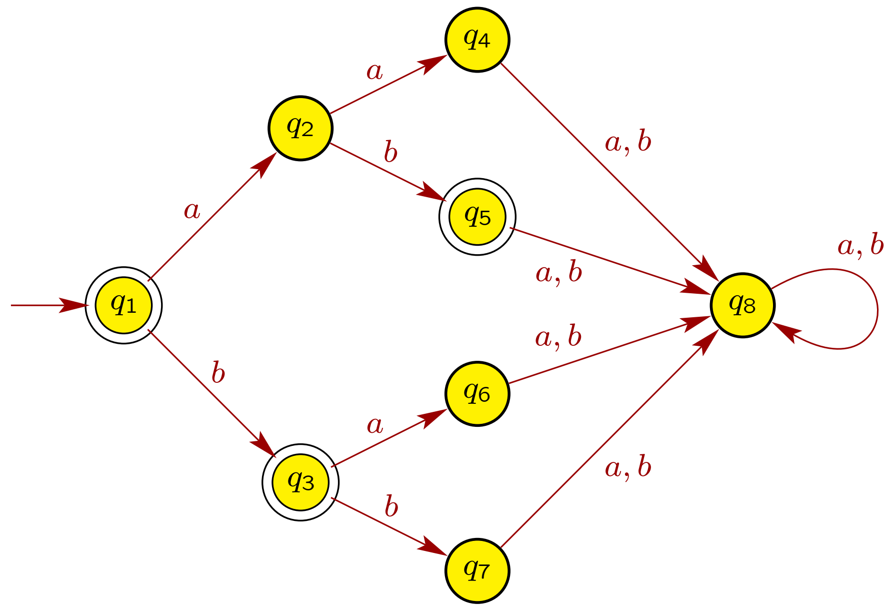
- $Q = \{q_1, q_2, q_3\}$
- $\Sigma = \{a, b\}$
- transition function δ is given by

	a	b
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_1	q_3

- q_1 is the start state
- $F = \{q_1, q_3\}$ is the set of accept states.

2. There are (infinitely) many correct DFAs for each part below.

   (a) The state diagram of one DFA that recognizes the language $A = \{\varepsilon, b, ab\}$ is below:



We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

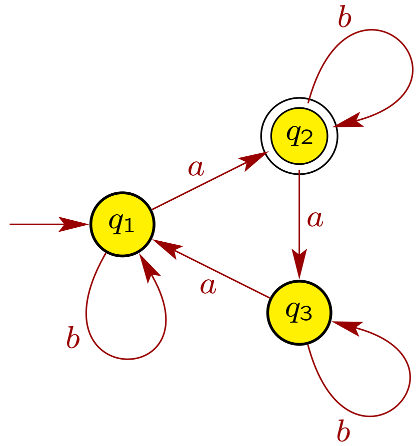
- $Q = \{q_1, q_2, \ldots, q_8\}$
- $\Sigma = \{a, b\}$
- transition function δ is given by

	a	b
q_1	q_2	q_3
q_2	q_4	q_5
q_3	q_6	q_7
q_4	q_8	q_8
q_5	q_8	q_8
q_6	q_8	q_8
q_7	q_8	q_8
q_8	q_8	q_8

- q_1 is the start state
- $F = \{q_1, q_3, q_5\}$ is the set of accept states.

There are simpler DFAs that recognize this language. Can you come up with one with only 4 states?

(b) The state diagram of one DFA that recognizes the language $B = \{\, w \in \Sigma^* \mid n_a(w) \bmod 3 = 1 \,\}$ is below:



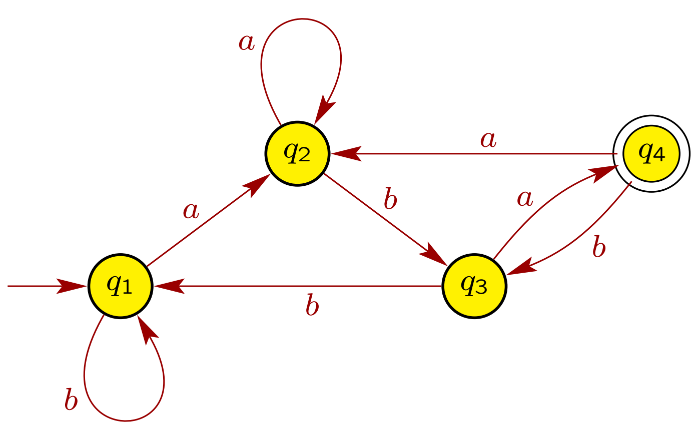
We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, q_3\}$
- $\Sigma = \{a, b\}$
- transition function δ is given by

	a	b
q_1	q_2	q_1
q_2	q_3	q_2
q_3	q_1	q_3

- q_1 is the start state
- $F = \{q_2\}$ is the set of accept states.

(c) The state diagram of one DFA that recognizes the language $C = \{\, w \in \Sigma^* \mid w = saba$ for some string $s \in \Sigma^* \,\}$ is below:



We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

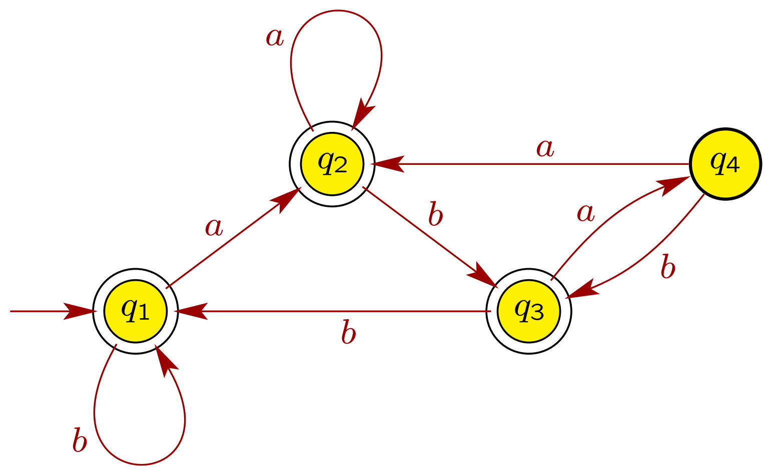
- $Q = \{q_1, q_2, q_3, q_4\}$
- $\Sigma = \{a, b\}$

- transition function δ is given by

	a	b
q_1	q_2	q_1
q_2	q_2	q_3
q_3	q_4	q_1
q_4	q_2	q_3

- q_1 is the start state
- $F = \{q_4\}$ is the set of accept states.

(d) Because $D = \overline{C}$, the complement of C, we can convert the DFA for C into a DFA for D by swapping the accept and non-accept states:



We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, q_3, q_4\}$
- $\Sigma = \{a, b\}$
- transition function δ is given by

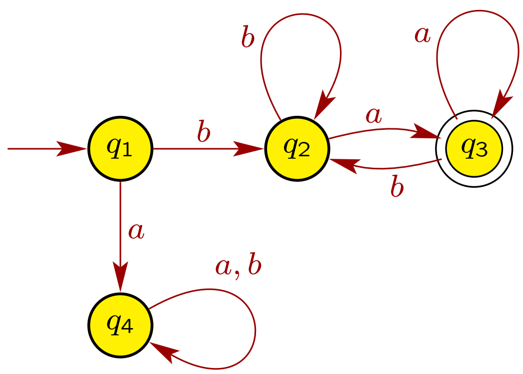
	a	b
q_1	q_2	q_1
q_2	q_2	q_3
q_3	q_4	q_1
q_4	q_2	q_3

- q_1 is the start state
- $F = \{q_1, q_2, q_3\}$ is the set of accept states.

(e) The state diagram of one DFA that recognizes the language

$$E = \{ w \in \Sigma^* \mid w \text{ begins with } b \text{ and ends with } a \}$$

is below:



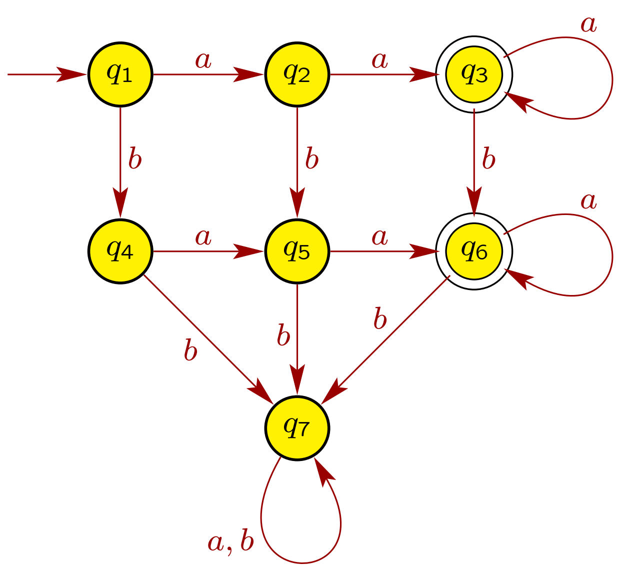
We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, q_3, q_4\}$
- $\Sigma = \{a, b\}$
- transition function δ is given by

	a	b
q_1	q_4	q_2
q_2	q_3	q_2
q_3	q_3	q_2
q_4	q_4	q_4

- q_1 is the start state
- $F = \{q_3\}$ is the set of accept states.

(f) The state diagram of one DFA that recognizes the language $F_0 = \{\, w \in \Sigma^* \mid n_a(w) \geq 2, n_b(w) \leq 1 \,\}$ is below:



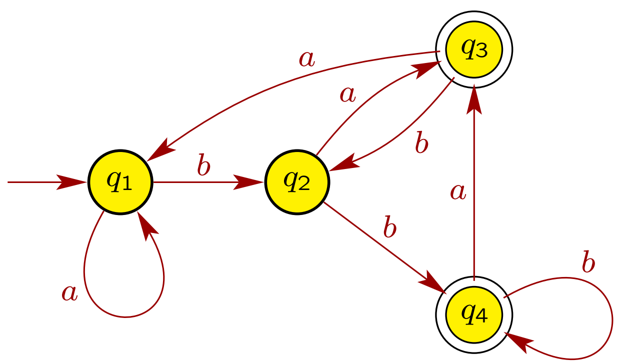
We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, \ldots, q_7\}$
- $\Sigma = \{a, b\}$
- transition function δ is given by

	a	b
q_1	q_2	q_4
q_2	q_3	q_5
q_3	q_3	q_6
q_4	q_5	q_7
q_5	q_6	q_7
q_6	q_6	q_7
q_7	q_7	q_7

- q_1 is the start state
- $F = \{q_3, q_6\}$ is the set of accept states.

(g) The state diagram of one DFA that recognizes the language $G = \{\, w \in \Sigma^* \mid |w| \geq 2, \text{ second-to-last symbol of } w \text{ is } b \,\}$ is below:



We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, q_3, q_4\}$
- $\Sigma = \{a, b\}$
- transition function δ is given by

	a	b
q_1	q_1	q_2
q_2	q_3	q_4
q_3	q_1	q_2
q_4	q_3	q_4

- q_1 is the start state
- $F = \{q_3, q_4\}$ is the set of accept states.

3. Show that, if M is a DFA that recognizes language B, swapping the accept and non-accept states in M yields a new DFA that recognizes $\overline{B}$, the complement of B. Conclude that the class of regular languages is closed under complement.

Answer:
Suppose language B over alphabet Σ has a DFA

$$M = (\, Q,\ \Sigma,\ \delta,\ q_1,\ F\,).$$

Then, a DFA for the complementary language $\overline{B}$ is

$$\overline{M} = (\, Q,\ \Sigma,\ \delta,\ q_1,\ Q - F\,).$$

The reason why $\overline{M}$ recognizes $\overline{B}$ is as follows. First note that M and $\overline{M}$ have the same transition function δ. Thus, since M is deterministic, $\overline{M}$ is also deterministic. Now consider any string $w \in \Sigma^*$. Running M on input string w will result in M ending in some state $r \in Q$. Since M is deterministic, there is only one possible state that M can end in on input w. If we run $\overline{M}$ on the same input w, then $\overline{M}$ will end in the same state r since M and $\overline{M}$ have the same transition function. Also, since $\overline{M}$ is deterministic, there is only one possible ending state that $\overline{M}$ can be in on input w.

Now suppose that $w \in B$. Then M will accept w, which means that the ending state $r \in F$, i.e., r is an accept state of M. But then $r \notin Q - F$, so $\overline{M}$ does not accept w since $\overline{M}$ has $Q - F$ as its set of accept states. Similarly, suppose that $w \notin B$. Then M will not accept w, which means that the ending state $r \notin F$. But then $r \in Q - F$, so $\overline{M}$ accepts w. Therefore, $\overline{M}$ accepts string w if and only M does not accept string w, so $\overline{M}$ recognizes language $\overline{B}$. Hence, the class of regular languages is closed under complement.

4. We say that a DFA M for a language A is *minimal* if there does not exist another DFA M' for A such that M' has strictly fewer states than M. Suppose that $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for A. Using M, we construct a DFA $\overline{M}$ for the complement $\overline{A}$ as $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$. Prove that $\overline{M}$ is a minimal DFA for $\overline{A}$.

   Answer:
   We prove this by contradiction. Suppose that $\overline{M}$ is not a minimal DFA for $\overline{A}$. Then there exists another DFA D for $\overline{A}$ such that D has strictly fewer states than $\overline{M}$. Now create another DFA D' by swapping the accepting and non-accepting states of D. Then D' recognizes the complement of $\overline{A}$. But the complement of $\overline{A}$ is just A, so D' recognizes A. Note that D' has the same number of states as D, and $\overline{M}$ has the same number of states as M. Thus, since we assumed that D has strictly fewer states than $\overline{M}$, then D' has strictly fewer states than M. But since D' recognizes A, this contradicts our assumption that M is a minimal DFA for A. Therefore, $\overline{M}$ is a minimal DFA for $\overline{A}$.

5. Give a formal proof that the class of regular languages is closed under intersection.

   Answer:
   Basic Idea: Recall that Theorem 1.25 establishes that the class of regular languages is closed under union. The approach that we will use to show that the class of regular

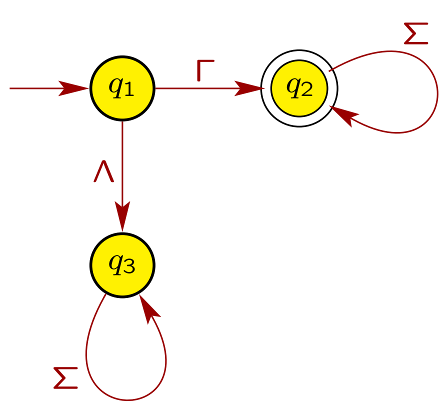
languages is closed under intersection is to modify the proof of Theorem 1.25. Specifically, Theorem 1.25 establishes that if A_1 is regular and A_2 is regular, then their union $A_1 \cup A_2$ is regular. The proof of Theorem 1.25 builds a DFA M_3' for $A_1 \cup A_2$ by simultaneously running a DFA M_1 for A_1 and a DFA M_2 for A_2, where the union DFA M_3' accepts if and only if M_1 accepts or M_2 accepts (or both accept). To instead build a DFA for the intersection $A_1 \cap A_2$, we can build a DFA M_3 by running M_1 and M_2 simultaneously, with the intersection DFA M_3 accepting if and only if both M_1 and M_2 accept. Below are the details.

Proof: Suppose A_1 and A_2 are defined over the same alphabet Σ. Suppose DFA M_1 recognizes A_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$. Suppose DFA M_2 recognizes A_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$. Define DFA $M_3 = (Q_3, \Sigma, \delta_3, q_3, F_3)$ for $A_1 \cap A_2$ as follows:

- Set of states of M_3 is
$$Q_3 = Q_1 \times Q_2 = \{ (x, y) \mid x \in Q_1,\ y \in Q_2 \}.$$
- The alphabet of M_3 is Σ.
- M_3 has transition function $\delta_3 : Q_3 \times \Sigma \to Q_3$ such that for $x \in Q_1$, $y \in Q_2$, and $\ell \in \Sigma$,
$$\delta_3(\,(x, y), \ell\,) \;=\; (\,\delta_1(x, \ell),\ \delta_2(y, \ell)\,)\,.$$
- The initial state of M_3 is $s_3 = (q_1, q_2) \in Q_3$.
- The set of accept states of M_3 is
$$F_3 \;=\; \{ (x, y) \in Q_1 \times Q_2 \mid x \in F_1 \text{ and } y \in F_2 \} = F_1 \times F_2.$$

Since $Q_3 = Q_1 \times Q_2$, the number of states in the new DFA M_3 is $|Q_3| = |Q_1| \cdot |Q_2|$. Thus, $|Q_3| < \infty$ since $|Q_1| < \infty$ and $|Q_2| < \infty$.

6. (a) Consider the language $L = \{ w \in \Sigma^* \mid w \text{ begins with a lower-case Roman letter} \}$ with $\Sigma = \{ \mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}, \mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots, \mathtt{9} \}$. Define $\Gamma = \{ \mathtt{a}, \mathtt{b}, \ldots, \mathtt{z} \}$ as the set of lower-case Roman letters, and $\Lambda = \{ \mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots, \mathtt{9} \}$ as the set of Arabic numerals, so $\Sigma = \Gamma \cup \Lambda$ with $\Gamma \cap \Lambda = \emptyset$. The state diagram of one DFA that recognizes L is below:



We formally express the DFA as a 5-tuple $(Q, \Sigma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, q_3\}$
- $\Sigma = \{\, \mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}, \mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots, \mathtt{9} \,\}$
- transition function δ is given by

	Γ	Λ
q_1	q_2	q_3
q_2	q_2	q_2
q_3	q_3	q_3

- q_1 is the start state
- $F = \{q_2\}$ is the set of accept states.

(b) Note that the string `if` $\in L$ but `if` $\notin J$ (because `if` is a reserved keyword), so $L \not\subseteq J$. Also, the string `AB` $\in J$ but `AB` $\notin L$, so $J \not\subseteq L$.