Homework 6 Solutions

1. Give pushdown automata that recognize the following languages. Give both a drawing and 6-tuple specification for each PDA.

(a) \( A = \{ w \in \{0, 1\}^* \mid w \text{ contains at least three } 1s \} \)

Answer:

We formally express the PDA as a 6-tuple \((Q, \Sigma, \Gamma, \delta, q_1, F)\), where

- \( Q = \{ q_1, q_2, q_3, q_4 \} \)
- \( \Sigma = \{0, 1\} \)
- \( \Gamma = \{0, 1\} \)
- transition function \( \delta : Q \times \Sigma \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma) \) is defined by

<table>
<thead>
<tr>
<th>Input:</th>
<th>0</th>
<th>1</th>
<th>( \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stack:</td>
<td>0</td>
<td>1</td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>{ (q_1, \varepsilon) }</td>
<td>{ (q_2, \varepsilon) }</td>
<td></td>
</tr>
<tr>
<td>( q_2 )</td>
<td>{ (q_2, \varepsilon) }</td>
<td>{ (q_3, \varepsilon) }</td>
<td></td>
</tr>
<tr>
<td>( q_3 )</td>
<td>{ (q_3, \varepsilon) }</td>
<td>{ (q_4, \varepsilon) }</td>
<td></td>
</tr>
<tr>
<td>( q_4 )</td>
<td>{ (q_4, \varepsilon) }</td>
<td>{ (q_4, \varepsilon) }</td>
<td></td>
</tr>
</tbody>
</table>

Blank entries are \( \emptyset \).

- \( q_1 \) is the start state
- \( F = \{ q_4 \} \)

Note that \( A \) is a regular language, so the language has a DFA. We can easily convert the DFA into a PDA by using the same states and transitions and never push nor pop anything to/from the stack.
(b) \( B = \{ w \in \{0, 1\}^* \mid w = w^R \text{ and the length of } w \text{ is odd} \} \)

Answer:

\[
\begin{array}{cccc}
q_1 & \rightarrow \varepsilon, \varepsilon \rightarrow \$$ & q_2 & 0, \varepsilon \rightarrow \varepsilon, 1, \varepsilon \rightarrow \varepsilon \\
q_2 & 0, \varepsilon \rightarrow 0 & q_3 & 0, 0 \rightarrow \varepsilon \\
q_3 & 1, \varepsilon \rightarrow 1 & q_4 & 1, 1 \rightarrow \varepsilon \\
\end{array}
\]

Since the length of any string \( w \in B \) is odd, \( w \) must have a symbol exactly in the middle position; i.e., \( |w| = 2n + 1 \) for some \( n \geq 0 \), and the \((n + 1)\)th symbol in \( w \) is the middle one. If a string \( w \) of length \( 2n + 1 \) satisfies \( w = w^R \), the first \( n \) symbols must match (in reverse order) the last \( n \) symbols, and the middle symbol doesn’t have to match anything. Thus, in the above PDA, the transition from \( q_2 \) to itself reads the first \( n \) symbols and pushes these on the stack. The transition from \( q_2 \) to \( q_3 \) nondeterministically identifies the middle symbol of \( w \), which doesn’t need to match any symbol, so the stack is unaltered. The transition from \( q_3 \) to itself then reads the last \( n \) symbols of \( w \), popping the stack at each step to make sure the symbols after the middle match (in reverse order) the symbols before the middle.

We formally express the PDA as a 6-tuple \((Q, \Sigma, \Gamma, \delta, q_1, F)\), where

- \( Q = \{q_1, q_2, q_3, q_4\} \)
- \( \Sigma = \{0, 1\} \)
- \( \Gamma = \{0, 1, \$$\} \) (use \$$\) to mark bottom of stack)
- transition function \( \delta : Q \times \Sigma \times \Gamma \varepsilon \rightarrow P(Q \times \Gamma \varepsilon) \) is defined by

<table>
<thead>
<tr>
<th>Input:</th>
<th>Stack: 0 1 $$ \varepsilon</th>
<th>Stack: 0 1 $$ \varepsilon</th>
<th>Stack: 0 1 $$ \varepsilon</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 )</td>
<td>( { (q_2, 0), (q_3, \varepsilon) } )</td>
<td>( { (q_2, 1), (q_3, \varepsilon) } )</td>
<td>( { (q_2, $$) } )</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>( { (q_3, \varepsilon) } )</td>
<td>( { (q_3, \varepsilon) } )</td>
<td>( { (q_3, \varepsilon) } )</td>
</tr>
<tr>
<td>( q_3 )</td>
<td>( { (q_3, \varepsilon) } )</td>
<td>( { (q_3, \varepsilon) } )</td>
<td>( { (q_4, \varepsilon) } )</td>
</tr>
<tr>
<td>( q_4 )</td>
<td>( { (q_3, \varepsilon) } )</td>
<td>( { (q_3, \varepsilon) } )</td>
<td>( { (q_4, \varepsilon) } )</td>
</tr>
</tbody>
</table>

Blank entries are \( \emptyset \).

- \( q_1 \) is the start state
- \( F = \{q_4\} \)

(c) \( C = \{ w \in \{0, 1\}^* \mid w = w^R \} \)

Answer:
The length of a string \( w \in C \) can be either even or odd. If it’s even, then there is no middle symbol in \( w \), so the first half of \( w \) is pushed on the stack, we move from \( q_2 \) to \( q_3 \) without reading, pushing, or popping anything, and then match the second half of \( w \) to the first half in reverse order by popping the stack. If the length of \( w \) is odd, then there is a middle symbol in \( w \), and the description of the PDA in part (b) applies.

We formally express the PDA as a 6-tuple \((Q, \Sigma, \Gamma, \delta, q_1, F)\), where

- \( Q = \{q_1, q_2, q_3, q_4\} \)
- \( \Sigma = \{0, 1\} \)
- \( \Gamma = \{0, 1, \$\} \) (use \$ to mark bottom of stack)
- transition function \( \delta : Q \times \Sigma \epsilon \times \Gamma \epsilon \rightarrow \mathcal{P}(Q \times \Gamma \epsilon) \) is defined by

<table>
<thead>
<tr>
<th>Input:</th>
<th>0</th>
<th>1</th>
<th>$</th>
<th>( \epsilon )</th>
<th>0</th>
<th>1</th>
<th>$</th>
<th>( \epsilon )</th>
<th>0</th>
<th>1</th>
<th>$</th>
<th>( \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stack:</td>
<td>(q_1)</td>
<td>(q_2)</td>
<td>(q_3)</td>
<td>(q_4)</td>
<td>(q_1)</td>
<td>(q_2)</td>
<td>(q_3)</td>
<td>(q_4)</td>
<td>(q_1)</td>
<td>(q_2)</td>
<td>(q_3)</td>
<td>(q_4)</td>
</tr>
<tr>
<td>0</td>
<td>( (q_1, \epsilon) )</td>
<td>( (q_2, 0), (q_3, \epsilon) )</td>
<td>( (q_3, \epsilon) )</td>
<td>( (q_2, $) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( (q_2, 0), (q_3, \epsilon) )</td>
<td>( (q_3, \epsilon) )</td>
<td>( (q_2, 1), (q_3, \epsilon) )</td>
<td>( (q_3, \epsilon) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>( (q_3, \epsilon) )</td>
<td>( (q_3, \epsilon) )</td>
<td>( (q_3, \epsilon) )</td>
<td>( (q_3, \epsilon) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Blank entries are \( \emptyset \).

- \( q_1 \) is the start state
- \( F = \{q_1, q_4\} \)

(d) \( D = \{a^i b^j c^k \mid i, j, k \geq 0, \text{ and } i = j \text{ or } j = k \} \)

**Answer:**
The PDA has a nondeterministic branch at \( q_1 \). If the string is \( a^i b^j c^k \) with \( i = j \), then the PDA takes the branch from \( q_1 \) to \( q_2 \). If the string is \( a^i b^j c^k \) with \( j = k \), then the PDA takes the branch from \( q_1 \) to \( q_5 \).

We formally express the PDA as a 6-tuple \((Q, \Sigma, \Gamma, \delta, q_1, F)\), where

- \( Q = \{q_1, q_2, \ldots, q_8\} \)
- \( \Sigma = \{a, b, c\} \)
- \( \Gamma = \{a, b, \$\} \) (use \$ to mark bottom of stack)
- transition function \( \delta : Q \times \Sigma \times \Gamma \epsilon \to P(Q \times \Gamma \epsilon) \) is defined by

Blank entries are \( \emptyset \).

- \( q_1 \) is the start state
- \( F = \{q_4, q_8\} \)

\((e)\ E = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i + j = k\} \)

Answer:

For every \( a \) and \( b \) read in the first part of the string, the PDA pushes an \( x \) onto the stack. Then it must read a \( c \) for each \( x \) popped off the stack.

We formally express the PDA as a 6-tuple \((Q, \Sigma, \Gamma, \delta, q_1, F)\), where

- \( Q = \{q_1, q_2, \ldots, q_5\} \)
- \( \Sigma = \{a, b, c\} \)
- \( \Gamma = \{x, \$\} \) (use \$ to mark bottom of stack)
- transition function \( \delta : Q \times \Sigma \times \Gamma \epsilon \to P(Q \times \Gamma \epsilon) \) is defined by
Blank entries are $\emptyset$.

- $q_1$ is the start state
- $F = \{q_5\}$

(f) $F = \{a^{2n}b^3n \mid n \geq 0\}$

Answer:

The PDA pushes a single $x$ onto the stack for every 2 $a$’s read at the beginning of the string. Then it pops a single $x$ for every 3 $b$’s read at the end of the string. We formally express the PDA as a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, \ldots, q_7\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{x, $\}$ (use $\$$ to mark bottom of stack)
- transition function $\delta : Q \times \Sigma \times \Gamma \times \varepsilon \rightarrow \mathcal{P}(Q \times \Gamma \times \varepsilon)$ is defined by

<table>
<thead>
<tr>
<th>Stack:</th>
<th>$a$</th>
<th>$b$</th>
<th>$\varepsilon$</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$x$</td>
<td>$$</td>
<td>$\varepsilon$</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_2$</td>
<td>$x$</td>
<td>$$</td>
<td>$\varepsilon$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(q_3, \varepsilon)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_3$</td>
<td>$q_2, x$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_4$</td>
<td></td>
<td></td>
<td>$(q_5, \varepsilon)$</td>
<td>$(q_7, \varepsilon)$</td>
</tr>
<tr>
<td>$q_5$</td>
<td></td>
<td></td>
<td></td>
<td>$(q_6, \varepsilon)$</td>
</tr>
<tr>
<td>$q_6$</td>
<td></td>
<td></td>
<td>$(q_4, \varepsilon)$</td>
<td></td>
</tr>
<tr>
<td>$q_7$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Blank entries are $\emptyset$.

- $q_1$ is the start state
- $F = \{q_7\}$

(g) $\emptyset$, with $\Sigma = \{0, 1\}$

Answer:
Because the PDA has no accept states, the PDA accepts no strings; i.e., the PDA recognizes the language $\emptyset$.

We formally express the PDA as a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_1, F)$, where

- $Q = \{q_1\}$
- $\Sigma = \{0, 1\}$
- $\Gamma = \{x\}$
- transition function $\delta : Q \times \Sigma \times \Gamma_{\epsilon} \rightarrow \mathcal{P}(Q \times \Gamma_{\epsilon})$ is defined by

<table>
<thead>
<tr>
<th>Input:</th>
<th>0</th>
<th>1</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stack:</td>
<td>$x$</td>
<td>$\epsilon$</td>
<td>$x$</td>
</tr>
</tbody>
</table>

Blank entries are $\emptyset$.

- $q_1$ is the start state
- $F = \emptyset$

(h) The language $H$ of strings of properly balanced left and right brackets: every left bracket can be paired with a unique subsequent right bracket, and every right bracket can be paired with a unique preceding left bracket. Moreover, the string between any such pair has the same property. For example, $[ ] [[[]]] [ ] \in A$.

Answer:

![Diagram](image)

We formally express the PDA as a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_1, F)$, where

- $Q = \{q_1, q_2, q_3\}$
- $\Sigma = \{[, ]\}$
- $\Gamma = \{[, $\}$ (use $\$$ to mark bottom of stack)
- transition function $\delta : Q \times \Sigma \times \Gamma_{\epsilon} \rightarrow \mathcal{P}(Q \times \Gamma_{\epsilon})$ is defined by

<table>
<thead>
<tr>
<th>Input:</th>
<th></th>
<th>$[ ]$</th>
<th>$\epsilon$</th>
<th></th>
<th></th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stack:</td>
<td>$[ $</td>
<td>$$$</td>
<td>$\epsilon$</td>
<td>$[ $</td>
<td>$$$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>$q_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_2$</td>
<td>${ (q_2, [ )}$</td>
<td>${ (q_2, \epsilon)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_3$</td>
<td>${ (q_3, \epsilon)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Blank entries are $\emptyset$.

- $q_1$ is the start state
- $F = \{q_3\}$

2. (a) Use the languages

\[
A = \{a^m b^n c^n \mid m, n \geq 0\} \quad \text{and} \\
B = \{a^n b^m c^n \mid m, n \geq 0\}
\]

Together with Example 2.36 of the textbook to show that the class of context-free languages is not closed under intersection.

**Answer:** The language $A$ is context free since it has CFG $G_1$ with rules

\[
S \rightarrow XY \\
X \rightarrow aX | \varepsilon \\
Y \rightarrow bYc | \varepsilon
\]

The language $B$ is context free since it has CFG $G_2$ with rules

\[
S \rightarrow XY \\
X \rightarrow aXb | \varepsilon \\
Y \rightarrow cY | \varepsilon
\]

But $A \cap B = \{a^n b^n c^n \mid n \geq 0\}$, which we know is not context free from Example 2.36 of the textbook. Thus, the class of context-free languages is not closed under intersection.

(b) Use part (a) and DeMorgan’s law (Theorem 0.20 of the textbook) to show that the class of context-free languages is not closed under complementation.

**Answer:** We will use a proof by contradiction, so we first assume the opposite of what we want to show; i.e., suppose the following is true:

**R1.** The class of context-free languages is closed under complementation.

Define the context-free languages $A$ and $B$ as in the previous part. Then R1 implies $\overline{A}$ and $\overline{B}$ are context-free. We know the class of context-free languages is closed under union, as shown on slide 2-101, so we then must have that $\overline{A} \cup \overline{B}$ is context-free. Then again apply R1 to conclude that $\overline{A} \cup \overline{B}$ is context-free. Now DeMorgan’s law states that $A \cap B = \overline{A} \cup \overline{B}$, but we showed in the previous part that $A \cap B$ is not context-free, which is a contradiction. Therefore, R1 must not be true.
3. Consider the following CFG $G = (V, \Sigma, R, S)$, where $V = \{ S, T, X \}$, $\Sigma = \{ a, b \}$, the start variable is $S$, and the rules $R$ are

$$
S \rightarrow aTXb \\
T \rightarrow XTS | \varepsilon \\
X \rightarrow a | b
$$

Convert $G$ to an equivalent PDA using the procedure given in Lemma 2.21.

**Answer:** First we create a PDA for $G$ that allows for pushing strings onto the stack:

Then we need to fix the non-compliant transitions, i.e., the ones for which a string of length more than 1 is pushed onto the stack. The only non-compliant transitions are the first two from $q_2$ back to itself, and the transition from $q_1$ to $q_2$. Fixing these gives the following PDA:
4. Use the pumping lemma to prove that the language $A = \{ 0^{2n} 1^{3n} 0^n \mid n \geq 0 \}$ is not context free.

**Answer:** Assume that $A$ is a CFL. Let $p$ be the pumping length of the pumping lemma for CFLs, and consider string $s = 0^{2p} 1^{3p} 0^p \in A$. Note that $|s| = 6p > p$, so the pumping lemma will hold. Thus, there exist strings $u, v, x, y, z$ such that $s = uvxyz = 0^{2p} 1^{3p} 0^p$, $uv^i x y^i z \in A$ for all $i \geq 0$, and $|vy| \geq 1$. We now consider all of the possible choices for $v$ and $y$:

- Suppose strings $v$ and $y$ are uniform (e.g., $v = 0^j$ for some $j \geq 0$, and $y = 1^k$ for some $k \geq 0$). Then $|vy| \geq 1$ implies that $j \geq 1$ or $k \geq 1$ (or both), so $uv^2 xy^2 z$ won’t have the correct number of 0’s at the beginning, 1’s in the middle, and 0’s at the end. Hence, $uv^2 xy^2 z \notin A$.
- Now suppose strings $v$ and $y$ are not both uniform. Then $uv^2 xy^2 z$ will not have the form $0 \cdots 01 \cdots 10 \cdots 0$. Hence, $uv^2 xy^2 z \notin A$.

Thus, there are no options for $v$ and $y$ such that $uv^i x y^i z \in A$ for all $i \geq 0$. This is a contradiction, so $A$ is not a CFL.

5. The Turing machine $M$ below recognizes the language $A = \{ 0^{2n} \mid n \geq 0 \}$.

In each of the parts below, give the sequence of configurations that $M$ enters when started on the indicated input string.

(a) 00

**Answer:** $q_1 00 \rightarrow \overline{q} 2 0 \rightarrow \overline{q} 3 \overline{u} \rightarrow \overline{q} 5 x \rightarrow q_5 x \rightarrow q_2 x \rightarrow q_2 x \rightarrow \overline{q} 2 \overline{u} \rightarrow \overline{q} accept$
(b) 000000

Answer: $q_1000000 \quad \underline{q_2}00000 \quad \underline{xq}_30000 \quad \underline{x}0q_4000$

$\underline{x}0xq_300 \quad \underline{x}0x0q_40 \quad \underline{x}0x0xq_3 \quad \underline{x}0x0q_5x \quad \underline{x}0xq_50x$

$\underline{x}0q_5x0x \quad \underline{xq}_50x0x \quad \underline{x}5x0x0x \quad q_5\underline{x}0x0x \quad q_2\underline{x}0x0x$

$\underline{x}q_20x0x \quad \underline{xxq}_3x0x \quad \underline{xxx}q_30x \quad \underline{xxx}0q_4x \quad \underline{xxx}0xq_4$