Homework 10 Solutions

1. If $A \leq_m B$ and $B$ is a regular language, does that imply that $A$ is a regular language?

**Answer:** No. For example, define the languages $A = \{0^n1^n \mid n \geq 0\}$ and $B = \{1\}$, both over the alphabet $\Sigma = \{0, 1\}$. Define the function $f : \Sigma^* \to \Sigma^*$ as

$$f(w) = \begin{cases} 1 & \text{if } w \in A, \\ 0 & \text{if } w \notin A. \end{cases}$$

Observe that $A$ is a context-free language, so it is also Turing-decidable. Thus, $f$ is a computable function. Also, $w \in A$ if and only if $f(w) = 1$, which is true if and only if $f(w) \in B$. Hence, $A \leq_m B$. Language $A$ is nonregular, but $B$ is regular since it is finite.

2. Show that $A_{TM}$ is not mapping reducible to $E_{TM}$. In other words, show that no computable function reduces $A_{TM}$ to $E_{TM}$. (Hint: Use a proof by contradiction, and facts you already know about $A_{TM}$ and $E_{TM}$.)

**Answer:** Suppose for a contradiction that $A_{TM} \leq_m E_{TM}$ via reduction $f$. This means that $w \in A_{TM}$ if and only if $f(w) \in E_{TM}$, which is equivalent to saying $w \notin A_{TM}$ if and only if $f(w) \notin E_{TM}$. Therefore, using the same reduction function $f$, we have that $A_{TM} \leq_m \overline{E_{TM}}$. However, $\overline{E_{TM}}$ is Turing-recognizable (HW 8, problem 4) and $\overline{A_{TM}}$ is not Turing-recognizable (Corollary 4.23), contradicting Theorem 5.22.

3. Consider the language

$$A_{\varepsilon_{TM}} = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \}.$$ 

Show that $A_{\varepsilon_{TM}}$ is undecidable.

**Answer:** We will show that $A_{TM}$ reduces to $A_{\varepsilon_{TM}}$. Suppose for contradiction that $A_{\varepsilon_{TM}}$ is decidable, and let $R$ be a TM that decides $A_{\varepsilon_{TM}}$. We construct another TM $S$ with input $\langle M, w \rangle$ that does the following. It first uses $M$ and $w$ to construct a new TM $M_2$, which takes input $x$. If $x \neq \varepsilon$, then $M_2$ accepts; otherwise, $M_2$ runs $M$ on input $w$ and $M_2$ accepts if $M$ accepts $w$. Note that $M_2$ recognizes the language $\Sigma^* - \{\varepsilon\}$ if $M$ rejects $w$; otherwise, $M_2$ recognizes the language $\Sigma^*$. In other words, $M_2$ accepts $\varepsilon$ if and only if $M$ accepts $w$. So our TM $S$ decides $A_{TM}$, which is a contradiction since we know $A_{TM}$ is undecidable.
Here are the details of our TM $S$:

\[ S = \text{“On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:} \]

1. Check if $\langle M, w \rangle$ is a valid encoding of a TM $M$ and string $w$.
   If not, reject.

2. Construct the following TM $M_2$:
   
   \[ M_2 = \text{“On input } x: \]
   
   1. If $x \neq \varepsilon$, accept.
   2. If $x = \varepsilon$, then run $M$ on input $w$ and accept if $M$ accepts $w$."

3. Run $R$ on input $\langle M_2 \rangle$.
4. If $R$ accepts, accept; if $R$ rejects, reject.”

4. A useless state in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a state in a Turing machine is useless. Formulate this problem as a language and show it is undecidable.

\textbf{Answer:} We define the problem as the language

\[ \text{USELESS}_{\text{TM}} = \{ \langle M, q \rangle \mid q \text{ is a useless state in TM } M \}. \]

We show that $\text{USELESS}_{\text{TM}}$ is undecidable by reducing $E_{\text{TM}}$ to $\text{USELESS}_{\text{TM}}$, where

\[ E_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}. \]

We know $E_{\text{TM}}$ is undecidable by Theorem 5.2.

Suppose that $\text{USELESS}_{\text{TM}}$ is decidable and that TM $R$ decides it. Note that for any Turing machine $M$ with accept state $q_{\text{accept}}$, $q_{\text{accept}}$ is useless if and only if $L(M) = \emptyset$. Thus, since TM $R$ solves $\text{USELESS}_{\text{TM}}$, we can use $R$ to check if $q_{\text{accept}}$ is a useless state to decide $E_{\text{TM}}$. Specifically, below is a TM $S$ that decides $E_{\text{TM}}$ by using the decider $R$ for $\text{USELESS}_{\text{TM}}$ as a subroutine:

\[ S = \text{“On input } \langle M \rangle, \text{ where } M \text{ is a TM:} \]

  1. Run TM $R$ on input $\langle M, q_{\text{accept}} \rangle$, where $q_{\text{accept}}$ is the accept state of $M$.
  2. If $R$ accepts, accept. If $R$ rejects, reject.”

However, since we known $E_{\text{TM}}$ is undecidable, there cannot exist a TM that decides $\text{USELESS}_{\text{TM}}$. 
