Homework 10 Solutions

1. If $A \leq_{\mathrm{m}} B$ and B is a regular language, does that imply that A is a regular language?

   Answer: No. For example, define the languages $A = \{\, 0^n 1^n \mid n \geq 0 \,\}$ and $B = \{1\}$, both over the alphabet $\Sigma = \{0, 1\}$. Define the function $f : \Sigma^* \to \Sigma^*$ as

   $$f(w) = \begin{cases} 1 & \text{if } w \in A, \\ 0 & \text{if } w \notin A. \end{cases}$$

   Observe that A is a context-free language, so it is also Turing-decidable. Thus, f is a computable function. Also, $w \in A$ if and only if $f(w) = 1$, which is true if and only if $f(w) \in B$. Hence, $A \leq_{\mathrm{m}} B$. Language A is nonregular, but B is regular since it is finite.

2. Show that A_{TM} is not mapping reducible to E_{TM}. In other words, show that no computable function reduces A_{TM} to E_{TM}. (Hint: Use a proof by contradiction, and facts you already know about A_{TM} and E_{TM}.)

   Answer: Suppose for a contradiction that $A_{\mathrm{TM}} \leq_{\mathrm{m}} E_{\mathrm{TM}}$ via reduction f. This means that $w \in A_{\mathrm{TM}}$ if and only if $f(w) \in E_{\mathrm{TM}}$, which is equivalent to saying $w \notin A_{\mathrm{TM}}$ if and only if $f(w) \notin E_{\mathrm{TM}}$. Therefore, using the same reduction function f, we have that $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} \overline{E_{\mathrm{TM}}}$. However, $\overline{E_{\mathrm{TM}}}$ is Turing-recognizable (HW 8, problem 4) and $\overline{A_{\mathrm{TM}}}$ is not Turing-recognizable (Corollary 4.23), contradicting Theorem 5.22.

3. Consider the language

   $$A\varepsilon_{\mathrm{TM}} = \{\, \langle M \rangle \mid M \text{ is a TM that accepts } \varepsilon \,\}.$$

   Show that $A\varepsilon_{\mathrm{TM}}$ is undecidable.

   Answer: We will show that A_{TM} reduces to $A\varepsilon_{\mathrm{TM}}$. Suppose for contradiction that $A\varepsilon_{\mathrm{TM}}$ is decidable, and let R be a TM that decides $A\varepsilon_{\mathrm{TM}}$. We construct another TM S with input $\langle M, w \rangle$ that does the following. It first uses M and w to construct a new TM M_2, which takes input x. If $x \neq \varepsilon$, then M_2 accepts; otherwise, M_2 runs M on input w and M_2 accepts if M accepts w. Note that M_2 recognizes the language $\Sigma^* - \{\varepsilon\}$ if M does not accept w; otherwise, M_2 recognizes the language Σ^*. In other words, M_2 accepts ε if and only if M accepts w. So our TM S decides A_{TM}, which is a contradiction since we know A_{TM} is undecidable.

Here are the details of our TM S:

S = "On input $\langle M, w \rangle$, where M is a TM and w is a string:

0. Check if $\langle M, w \rangle$ is a valid encoding of a TM M and string w. If not, *reject*.

1. Construct the following TM M_2 from M and w:

M_2 = "On input x:

1. If $x \neq \varepsilon$, *accept*.

2. If $x = \varepsilon$, then run M on input w and *accept* if M accepts w."

2. Run R on input $\langle M_2 \rangle$.

3. If R accepts, *accept*; if R rejects, *reject*."

4. A *useless state* in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a state in a Turing machine is useless. Formulate this problem as a language and show it is undecidable.

Answer: We define the decision problem with the language

$$USELESS_{\rm TM} = \{ \langle M, q \rangle \mid q \text{ is a useless state in TM } M \}.$$

We show that $USELESS_{\rm TM}$ is undecidable by reducing $E_{\rm TM}$ to $USELESS_{\rm TM}$, where $E_{\rm TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$. We know $E_{\rm TM}$ is undecidable by Theorem 5.2.

Suppose that $USELESS_{\rm TM}$ is decidable and that TM R decides it. Note that for any Turing machine M with accept state $q_{\rm accept}$, $q_{\rm accept}$ is useless if and only if $L(M) = \emptyset$. Thus, because TM R solves $USELESS_{\rm TM}$, we can use R to check if $q_{\rm accept}$ is a useless state to decide $E_{\rm TM}$. Specifically, below is a TM S that decides $E_{\rm TM}$ by using the decider R for $USELESS_{\rm TM}$ as a subroutine:

S = "On input $\langle M \rangle$, where M is a TM:

1. Run TM R on input $\langle M, q_{\rm accept} \rangle$, where $q_{\rm accept}$ is the accept state of M.

2. If R accepts, *accept*. If R rejects, *reject*."

However, because we known $E_{\rm TM}$ is undecidable, there cannot exist a TM that decides $USELESS_{\rm TM}$.