CS 341: Foundations of CS II

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Chapter 5
Reducibility

Contents

Introduction

- Previously, we saw
  - Church-Turing Thesis
  - Many problems are solvable using TMs
  - One problem (language), A_{TM}, is unsolvable by TMs, where

$$A_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts string } w \}$$

- We now will see many other computationally unsolvable problems.
- We will do this by using **reductions**.
- **Example:**

Finding your way around a city
reduces to
obtaining a city map.

Reducibility

- Reduction always involves two problems (languages), A and B.
- **Definition:** If A **reduces** to B, then can use any solution of B to solve A.
- **Remarks:**
  - We showed that A_{NFA} is decidable by reducing A_{NFA} to A_{DFA}.
  - Definition of reduction says nothing about solving A or B alone.
  - If A is reducible to B, then A cannot be harder than B.
  - The statement "$p \Rightarrow q$" is equivalent to "$\neg q \Rightarrow \neg p$".
  - Suppose A reduces to B. Then
    - If I can solve B, then I can solve A.
    - Equivalently, if I can't solve A, then I can't solve B.
    - Equivalently, if A is undecidable, then B is undecidable.

Reducibility

- It required some effort to prove that A_{TM} is not decidable.
- But now we can build on this result as follows:
  - To show another language L is undecidable, we typically show A_{TM} reduces to L.
  - If "language L is decidable" implies "A_{TM} is decidable," then L is not decidable.
- Typical approach to show L is undecidable via reduction from A_{TM} to L
  - Suppose L is decidable.
  - Let R be a TM that decides L.
  - Using R as subroutine,
    - can construct another TM S that decides A_{TM}.
  - But A_{TM} is not decidable.
  - Conclusion: L is not decidable.

Halting Problem for TMs is Undecidable

- Recall that A_{TM} (acceptance problem for TMs) is undecidable, where

$$A_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is TM that } \textbf{accepts} \text{ string } w \}.$$

- Another decision problem: Does TM M **halt** on input w?

$$HALT_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is TM that } \textbf{halts} \text{ on string } w \}.$$

- In this case (but not others), A_{TM} and $HALT_{\mathsf{TM}}$ have same universe

$$\Omega = \{ \langle M, w \rangle \mid M \text{ is TM, } w \text{ is string} \}.$$

- Given $\langle M, w \rangle \in \Omega$ of specific pair of TM M and string w,
  - if M halts on input w, then $\langle M, w \rangle \in HALT_{\mathsf{TM}}$,
  - if M doesn't halt on input w, then $\langle M, w \rangle \notin HALT_{\mathsf{TM}}$.
- How does $HALT_{\mathsf{TM}}$ differ from A_{TM}?

Theorem 5.1
$HALT_{\mathsf{TM}}$ is undecidable.

Basic Idea of Proof that $HALT_{\mathsf{TM}}$ is Undecidable

$$A_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \textbf{ accepts} \text{ string } w \},$$
$$HALT_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \textbf{ halts} \text{ on string } w \}.$$

Basic idea of proof by contradiction: reduce A_{TM} to $HALT_{\mathsf{TM}}$

- Suppose $\exists$ TM R that **decides** $HALT_{\mathsf{TM}}$.
- How could we use R to construct TM to **decide** A_{TM}?
- Recall universal TM U **recognizes** A_{TM}:

$U =$ "On input $\langle M, w \rangle \in \Omega$, where M is a TM and w is a string:
1. Simulate M on input w.
2. If M ever enters its accept state, *accept*;
if M ever enters its reject state, *reject*."

- U **doesn't decide** A_{TM} since M may loop on w in stage 1.
- **Solution:** first run R on $\langle M, w \rangle$ to see if it's safe to run M on w.

Proof that $HALT_{\mathsf{TM}}$ is Undecidable

$$A_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \textbf{ accepts} \text{ string } w \},$$
$$HALT_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \textbf{ halts} \text{ on string } w \}.$$

- Assume $\exists$ TM R that decides $HALT_{\mathsf{TM}}$.
- Define TM S to decide A_{TM} using TM R as follows:

$S =$ "On input $\langle M, w \rangle \in \Omega$, where M is a TM and w a string:
1. Run R on input $\langle M, w \rangle$.
2. If R rejects, *reject*.
3. If R accepts, simulate M on input w until it halts.
4. If M accepts, *accept*; otherwise, *reject*."

- TM S always halts and decides A_{TM}
  - S accepts $\langle M, w \rangle \in A_{\mathsf{TM}}$, and S rejects $\langle M, w \rangle \notin A_{\mathsf{TM}}$.
- Thus, deciding A_{TM} is reduced to deciding $HALT_{\mathsf{TM}}$.
- But A_{TM} is undecidable, so $HALT_{\mathsf{TM}}$ must also be undecidable.

Emptiness Problem for TMs is Undecidable

- **Decision problem:** Does a TM M recognize the empty language?

$$\begin{aligned} E_{\mathsf{TM}} &= \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \} \\ &\subseteq \{ \langle M \rangle \mid M \text{ is a TM} \} \equiv \Omega_E, \end{aligned}$$

  where universe Ω_E comprises all TMs.

- For a specific encoded TM $\langle M \rangle \in \Omega_E$,
  - if M accepts at least one string, then $\langle M \rangle \notin E_{\mathsf{TM}}$,
  - if M accepts no strings, then $\langle M \rangle \in E_{\mathsf{TM}}$.

Theorem 5.2
E_{TM} is undecidable.

Proof Idea: Reduce A_{TM} to E_{TM}.

- Suppose E_{TM} is decidable.
- Let R be a TM that decides E_{TM}.
- Use TM R to construct another TM S that decides A_{TM}.
- But since A_{TM} is undecidable, E_{TM} must also be.

Constructing Decider S for A_{TM} From Decider R for E_{TM}

$$E_{\mathsf{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$$

- **Bad Idea:** When S receives input $\langle M, w \rangle$, it calls R with input $\langle M \rangle$.
  - If R accepts, then $L(M) = \emptyset$.
    - In particular, M does not accept w, so S *rejects* input $\langle M, w \rangle$.
  - If R rejects, then $L(M) \neq \emptyset$, so M accepts at least one string.
    - But don't know if M accepts w, so TM S can't decide A_{TM}.
- **Fix:** Create another TM M_1 from TM M and w as follows:

  $M_1 =$ "On input x:
  1. If $x \neq w$, *reject*.
  2. If $x = w$, run M on input w, and *accept* iff M accepts."

  - w is only string M_1 could accept, so one of 2 cases occurs:
    - If $\langle M, w \rangle \in A_{\mathsf{TM}}$, then $L(M_1) = \{w\}$, so $\langle M_1 \rangle \notin E_{\mathsf{TM}}$.
    - If $\langle M, w \rangle \notin A_{\mathsf{TM}}$, then $L(M_1) = \emptyset$, so $\langle M_1 \rangle \in E_{\mathsf{TM}}$.

Proof: $E_{\mathsf{TM}} = \{ \langle M \rangle \mid M$ is TM and $L(M) = \emptyset \}$ is Undecidable

- Reduce A_{TM} to E_{TM}: suppose $\exists$ TM R that decides E_{TM}.
- Define TM S to decide A_{TM} using decider R for E_{TM} as follows:

  $S =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string:
  1. Construct TM M_1 from M and w as follows:

     $M_1 =$ "On input x:
     (1) If $x \neq w$, *reject*.
     (2) If $x = w$, run M on input w, and *accept* iff M accepts."
  2. Run R on input $\langle M_1 \rangle$.
  3. If R accepts, *reject*; if R rejects, *accept*."

- Note that

$$\begin{aligned} \langle M_1 \rangle \notin E_{\mathsf{TM}} \iff & L(M_1) \neq \emptyset \iff M \text{ accepts } w \\ \iff & \langle M, w \rangle \in A_{\mathsf{TM}}. \end{aligned}$$

- But then TM S decides A_{TM}, which is undecidable.
- Therefore, TM R cannot exist, so E_{TM} is undecidable.

TM Recognizing Regular Language is Undecidable

- **Decision problem:** Does a TM M recognize a regular language?

$$\begin{aligned} REG_{\mathsf{TM}} &= \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language} \} \\ &\subseteq \{ \langle M \rangle \mid M \text{ is a TM} \} \equiv \Omega_{REG}, \end{aligned}$$

  where universe Ω_{REG} comprises all TMs.

- For a specific encoded TM $\langle M \rangle \in \Omega_{REG}$,
  - if $L(M)$ is regular, then $\langle M \rangle \in REG_{\mathsf{TM}}$,
  - if $L(M)$ is nonregular, then $\langle M \rangle \notin REG_{\mathsf{TM}}$.

Theorem 5.3
REG_{TM} is undecidable.

Proof Idea: Reduce A_{TM} to REG_{TM}.

- Assume REG_{TM} is decidable.
- Let R be a TM that decides REG_{TM}.
- Use TM R to construct TM S that decides A_{TM}.
- But how do we do this?

Constructing Decider S for A_{TM} from Decider R for REG_{TM}

$$REG_{\mathsf{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language} \}.$$

- TM S is given input $\langle M, w \rangle$.
- TM S first constructs a TM M' using $\langle M, w \rangle$ so that $L(M')$ is a regular language if and only if M accepts w.
  - If M does not accept w, then M' recognizes language
    $$\{ 0^n 1^n \mid n \geq 0 \},$$
    which is **nonregular**.
  - If M accepts w, then M' recognizes language Σ^*, which is **regular**.
- We construct M' as follows:
  - M' automatically accepts all strings in $\{ 0^n 1^n \mid n \geq 0 \}$.
  - In addition, if M accepts w, then M' accepts all other strings.

Proof that REG_{TM} is Undecidable

- Suppose that REG_{TM} is decidable.
- Let R be a TM that decides REG_{TM}.
- Use R to construct TM S to decide A_{TM}:

  S = "On input $\langle M, w \rangle$, where M is a TM and w is a string:

  1. Construct following TM M' from M and w:

     M' = "On input x:
     1. If $x \in \{ 0^n 1^n \mid n \geq 0 \}$, *accept*.
     2. If $x \notin \{ 0^n 1^n \mid n \geq 0 \}$, run M on input w and *accept* iff M accepts w."
  2. Run R on input $\langle M' \rangle$.
  3. If R accepts, *accept*; if R rejects, *reject*."
- $\langle M' \rangle \in REG_{\mathsf{TM}} \iff \langle M, w \rangle \in A_{\mathsf{TM}}$, so S decides A_{TM}, which is impossible.

Equivalence of 2 TMs is Undecidable

- **Decision problem:** Do 2 TMs recognize the same language?

  $$\begin{aligned} EQ_{\mathsf{TM}} &= \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \} \\ &\subseteq \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs} \} \equiv \Omega_{EQ}, \end{aligned}$$

  where universe Ω_{EQ} comprises all pairs of TMs.
- For any specific encoded pair $\langle M_1, M_2 \rangle \in \Omega_{EQ}$,
  - if $L(M_1) = L(M_2)$, then $\langle M_1, M_2 \rangle \in EQ_{\mathsf{TM}}$,
  - if $L(M_1) \neq L(M_2)$, then $\langle M_1, M_2 \rangle \notin EQ_{\mathsf{TM}}$.

Theorem 5.4
EQ_{TM} is undecidable.

Proof that EQ_{TM} is Undecidable

- Recall

  $$EQ_{\mathsf{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2) \}.$$
- Reduce E_{TM} to EQ_{TM} as follows:
  - Let $M_2 = M_\emptyset$ be a TM with $L(M_\emptyset) = \emptyset$.
  - A TM that decides EQ_{TM} can also decide E_{TM} by deciding if $\langle M_1, M_\emptyset \rangle \in EQ_{\mathsf{TM}}$.
    - $\langle M_1 \rangle \in E_{\mathsf{TM}} \iff \langle M_1, M_\emptyset \rangle \in EQ_{\mathsf{TM}}$
- Since E_{TM} is undecidable (Theorem 5.2), EQ_{TM} must be undecidable.
- We'll see later that EQ_{TM} is
  - not Turing-recognizable
  - not co-Turing-recognizable

Other Undecidable Problems

- Does a TM recognize a finite language?
- Does a TM recognize a context-free language?
- Does a TM recognize a decidable language?
- Does a TM halt on all inputs?
- Does a TM have a state that is never entered on any input string?

Rice's Theorem.

- *Informally:* Every non-trivial property $\mathcal{P}$ of languages of Turing machines is undecidable.
- *Formally:* Let $\mathcal{P}$ be a language consisting of TM descriptions such that
  1. $\mathcal{P}$ contains some, but not all, TM descriptions, and
  2. whenever $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in \mathcal{P}$ iff $\langle M_2 \rangle \in \mathcal{P}$.

  Then $\mathcal{P}$ is undecidable.

Proof of Rice's Theorem: Reduce A_{TM} to $\mathcal{P}$

- Suppose $\mathcal{P}$ is decided by TM $R_{\mathcal{P}}$.
- Let $T_\emptyset$ be a TM that always rejects, so $L(T_\emptyset) = \emptyset$.
- Without loss of generality, assume $\langle T_\emptyset \rangle \not\in \mathcal{P}$. (Otherwise, consider $\overline{\mathcal{P}}$.)
- Because we assumed $\mathcal{P}$ is nontrivial, $\exists$ TM T with $\langle T \rangle \in \mathcal{P}$.
- Now design TM S to decide A_{TM} using $R_{\mathcal{P}}$'s ability to distinguish between $T_\emptyset$ and T.

$S =$ "On input $\langle M, w \rangle$, where M is a TM and w a string:

1. Use M and w to construct the following TM M_w:

$M_w =$ "On input x:

1. Simulate M on input w. If it halts and rejects, *reject*.

2. Simulate T on input x. If it accepts, *accept*."

2. Use TM $R_{\mathcal{P}}$ to determine whether $\langle M_w \rangle \in \mathcal{P}$.
If YES, *accept*. If NO, *reject*."

Proof of Rice's Theorem: Reduce A_{TM} to $\mathcal{P}$ (cont.)

- Note that TM M_w simulates T if M accepts w.
- Hence,
  - $L(M_w) = L(T)$ if M accepts w,
  - $L(M_w) = \emptyset$ if M does not accept w.
- Therefore, $\langle M_w \rangle \in \mathcal{P}$ iff M accepts w.
- Hence, S decides A_{TM}, which is impossible since A_{TM} is undecidable.
- Thus, $\mathcal{P}$ is undecidable.

Limited Success Thus Far

- Our reductions have been straightforward:
  - Transform TM for some language into a similar TM that decides another language
- As a result, the languages we proved are undecidable are similar:
  - A_{TM}, EQ_{TM}, $HALT_{\mathsf{TM}}$, etc.
- For languages concerning questions not about TMs, we have to use a different approach.
  - e.g., Hilbert's 10th problem
- Recall interpretation of TM configuration:

$$1011q_7 01$$

  - current state is q_7
  - LHS of tape is 1011, and RHS of tape is 01
  - tape head is on RHS 0

Computation Histories

Definition: An **accepting computation history** for a TM M on a string w is a sequence of configurations

$$C_1, C_2, \ldots, C_k$$

for some $k \geq 1$ such that the following properties hold:

1. C_1 is the start configuration of M on w.
2. Each C_j yields C_{j+1}.
3. C_k is an accepting configuration.

Definition: A **rejecting computation history** for M on w is the same except last configuration C_k is a rejecting configuration of M.

Remarks About Computation Histories

- Accepting and rejecting computation histories are finite.
- If M does not halt on w,
  - then no accepting or rejecting computation history exists.
- Useful for both
  - deterministic TMs (one history)
  - nondeterministic TMs (many histories).
- "$\langle M, w\rangle \notin A_{\text{TM}}$" is equivalent to
  - "$\nexists$ accepting computation history $C_1, \ldots, C_k$ for M on w"
  - "All histories $C_1, \ldots, C_k$ are non-accepting ones for M on w".

Context-Free Languages

Decision problem: Does a CFG generate all strings over Σ?

$$\begin{aligned} ALL_{\text{CFG}} &= \{ \langle G\rangle \mid G \text{ is CFG with } L(G) = \Sigma^* \} \\ &\subseteq \{ \langle G\rangle \mid G \text{ is CFG} \} \equiv \Omega_{ALLC}. \end{aligned}$$

Theorem 5.13
ALL_{CFG} is undecidable.

Proof Idea: (see Sipser for full proof)

- Approach: Reduce A_{TM} to ALL_{CFG}.
- Construct a CFG G from TM M and input w.
  - If M does not accept w, then G generates all strings.
  - If M accepts w, then G generates all strings **except** the accepting computation histories for M on w.
- CFG G generates all strings iff TM M does not accept w.

Mapping Reducibility

- Thus far, we have seen several ways to reduce one problem to another.
- Reductions appear in
  - decidability theory
  - complexity theory (as we'll see later in Chapter 7).
- Now we want to formalize the notion of reducibility.

Computable Functions

- Suppose we have 2 languages A and B, where
  - A defined over alphabet Σ_1, so $A \subseteq \Sigma_1^*$, i.e., universe $\Omega_1 = \Sigma_1^*$
  - B defined over alphabet Σ_2, so $B \subseteq \Sigma_2^*$, i.e., universe $\Omega_2 = \Sigma_2^*$
- Informally speaking, A is reducible to B if we can use a "black box" for B to build an algorithm for A.
- **Definition:** A function

$$f : \Sigma_1^* \to \Sigma_2^*$$

is a **computable function** if some TM M, on every input $w \in \Sigma_1^*$, halts with just $f(w) \in \Sigma_2^*$ on its tape.

- All the usual integer computations are computable:
  - Addition, multiplication, sorting, etc.

Computable Functions

One useful class of computable functions transforms one TM into another.

Example:

$T =$ "On input w:

1. If $w = \langle M \rangle$, where M is some TM,
   - Construct $\langle M' \rangle$, where M' is a TM such that
     - $L(M') = L(M)$, but
     - M' never tries to move tape head off LHS of tape."

The function T accomplishes this by adding several states to the description of M.

Mapping Reducibility

Definition: Suppose

- A is defined over alphabet Σ_1, so $A \subseteq \Sigma_1^*$, i.e., universe $\Omega_1 = \Sigma_1^*$
- B is defined over alphabet Σ_2, so $B \subseteq \Sigma_2^*$, i.e., universe $\Omega_2 = \Sigma_2^*$

Then A is **mapping reducible** to B, written

$$A \leq_{\mathrm{m}} B$$

if there is a computable function

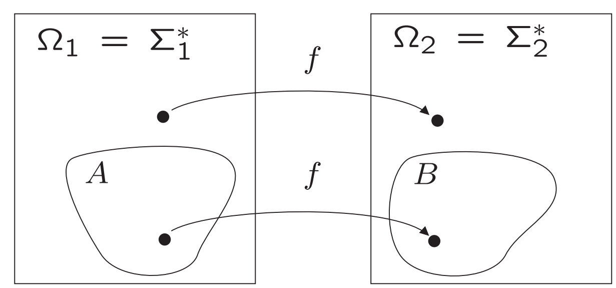
$$f : \Sigma_1^* \to \Sigma_2^*$$

such that, for every $w \in \Sigma_1^*$,

$$w \in A \iff f(w) \in B.$$

The function f is called a **reduction** of A to B.
(f is also called a **many-one reduction**.)

Language A is Mapping Reducible to B



$$w \in A \iff f(w) \in B$$

YES instance for problem A $\iff$ YES instance for problem B

- Consider decision problems of membership for A and B:
  - Does instance from Ω_1 belong to A?
  - Does instance from Ω_2 belong to B?
- If $A \leq_{\mathrm{m}} B$ and can solve membership problem for B, then can solve membership problem for A.

Example: Mapping Reduction $A_{\mathsf{TM}} \leq_{\mathsf{m}} HALT_{\mathsf{TM}}$

- Recall that

$$A_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is TM that } \textbf{accepts} \text{ string } w \} \subseteq \Omega_A,$$
$$HALT_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is TM that } \textbf{halts} \text{ on string } w \} \subseteq \Omega_H.$$

- In this case (but not always), same universes $\Omega_A = \Omega_H = \Omega$, with

$$\Omega = \{ \langle M, w \rangle \mid M \text{ is TM, } w \text{ is string} \}$$

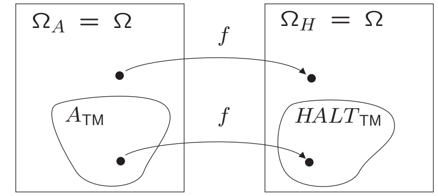
- We previously proved that $HALT_{\mathsf{TM}}$ is undecidable by showing A_{TM} reduces to $HALT_{\mathsf{TM}}$.
- To show $A_{\mathsf{TM}} \leq_{\mathsf{m}} HALT_{\mathsf{TM}}$, need function $f : \Omega_A \to \Omega_H$, with
  - input $\langle M, w \rangle \in \Omega_A$ is instance for acceptance problem for TMs
  - output $f(\langle M, w \rangle) = \langle M', w' \rangle \in \Omega_H$ is instance for halting problem for TMs
  - $\langle M, w \rangle \in A_{\mathsf{TM}} \iff f(\langle M, w \rangle) = \langle M', w' \rangle \in HALT_{\mathsf{TM}}$.

Example: Mapping Reduction $A_{\mathsf{TM}} \leq_{\mathsf{m}} HALT_{\mathsf{TM}}$

- Recall $\Omega_A = \Omega_H = \Omega$, with

  $\Omega = \{ \langle M, w \rangle \mid \text{TM } M, \text{ string } w \}$

- TM F computes reducing fcn f



$F =$ "On input $\langle M, w \rangle \in \Omega_A$, where M is TM and w is string:

1. Construct the following TM M':

   $M' =$ "On input x:
   (1) Run M on input x.
   (2) If M accepts, $accept$.
   (3) If M rejects, enter a loop."

2. Output $\langle M', w \rangle \in \Omega_H$."

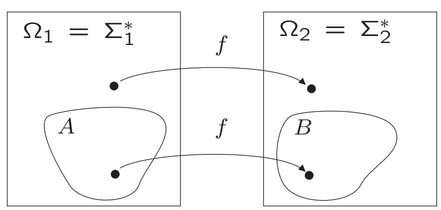
- Note that $\langle M, w \rangle \in A_{\mathsf{TM}} \iff \langle M', w \rangle \in HALT_{\mathsf{TM}}$.

Decidability obeys $\leq_{\mathsf{m}}$ Ordering

Theorem 5.22

If $A \leq_{\mathsf{m}} B$ and B is decidable, then A is decidable.

Proof.



- Let M_B be TM that decides B.
- Let $f : \Sigma_1^* \to \Sigma_2^*$ be reducing fcn from A to B.
- Consider the following TM:

  $M_A =$ "On input $w \in \Sigma_1^*$:
  1. Compute $f(w) \in \Sigma_2^*$.
  2. Run M_B on input $f(w)$ and give the same result."

- Since f is reducing function, $w \in A \iff f(w) \in B$.
  - If $w \in A$, then $f(w) \in B$, so M_B and M_A accept.
  - If $w \notin A$, then $f(w) \notin B$, so M_B and M_A reject.
- Thus, M_A decides A.

Undecidability obeys $\leq_{\mathsf{m}}$ Ordering

Corollary 5.23

If $A \leq_{\mathsf{m}} B$ and A is undecidable, then B is undecidable also.

Proof. Language A undecidable and B decidable contradicts the previous theorem.

Recall: Complements $\overline{A} = \Sigma_1^* - A$ and $\overline{B} = \Sigma_2^* - B$.

Fact: If $A \leq_{\mathsf{m}} B$, then $\overline{A} \leq_{\mathsf{m}} \overline{B}$.

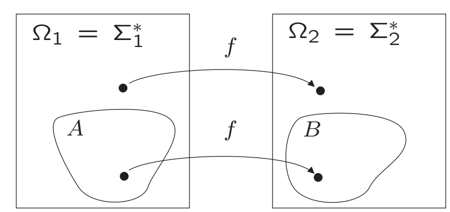
Proof.

- Let f be reducing fcn of A to B:

$$w \in A \iff f(w) \in B.$$

- Same fcn f shows $\overline{A} \leq_{\mathsf{m}} \overline{B}$ since

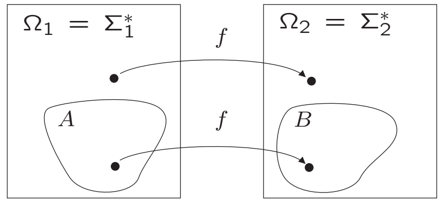
$$w \in \overline{A} \iff f(w) \in \overline{B}.$$



Recognizability and $\leq_m$

Theorem 5.28
If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

Proof.



- Let M_B be TM recognizing B.
- Let f be reducing fcn from A to B.
- Define a new TM as follows:
  $M_A =$ "On input $w \in \Sigma_1^*$:
  1. Compute $f(w) \in \Sigma_2^*$.
  2. Run M_B on input $f(w)$ and give the same result."
- Since f is a reducing function, $w \in A \iff f(w) \in B$.
  - If $w \in A$, then $f(w) \in B$, so M_B and M_A accept.
  - If $w \notin A$, then $f(w) \notin B$, so M_B and M_A reject or loop.
- Thus, M_A recognizes A.

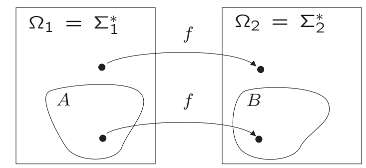
Unrecognizability and $\leq_m$

Corollary 5.29
If $A \leq_m B$ and A is not Turing-recognizable,
then B is not Turing-recognizable.

Proof. Language A not Turing-recognizable and B Turing-recognizable contradicts the previous theorem.

Fact: If $A \leq_m B$ and A is not co-Turing-recognizable,
then B is not co-Turing-recognizable.

Proof.



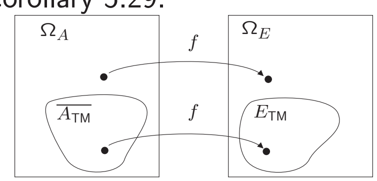
- If A is not co-Turing-recognizable,
  then complement $\overline{A}$ is not Turing-recog.
- $A \leq_m B$ implies $\overline{A} \leq_m \overline{B}$ (see slide 5-32).
- $\overline{B}$ is not Turing-recog. (Corollary 5.29).
- Hence, B is not co-Turing-recognizable.

E_{TM} is not Turing-recognizable

Recall: the emptiness problem for TMs:

$$E_{TM} = \{ \langle M \rangle \mid M \text{ is TM with } L(M) = \emptyset \}$$
$$\subseteq \{ \langle M \rangle \mid M \text{ is TM} \} \equiv \Omega_E$$

Proof. Reduce $\overline{A_{TM}} \leq_m E_{TM}$, and apply Corollary 5.29.



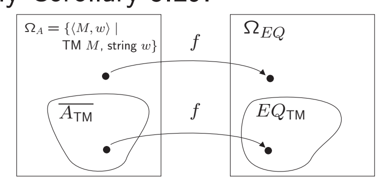
- $\Omega_A = \{ \langle M, w \rangle \mid \text{TM } M, \text{ string } w \}$
- Reducing fcn $f(\langle M, w \rangle) = \langle M' \rangle$,
  where M' is following TM:

  $M' =$ "On input x:
  1. Ignore input x, and run M on input w.
  2. If M accepts w, *accept*; if M rejects w, *reject*."
- If M accepts w (i.e., $\langle M, w \rangle \notin \overline{A_{TM}}$), then $L(M') = \Sigma^*$;
  if M doesn't accept w (i.e., $\langle M, w \rangle \in \overline{A_{TM}}$), then $L(M') = \emptyset$.
- Thus, $\langle M, w \rangle \in \overline{A_{TM}} \iff f(\langle M, w \rangle) = \langle M' \rangle \in E_{TM}$.
- Cor. 5.29 implies E_{TM} not TM-recog. since $\overline{A_{TM}}$ also isn't (Cor. 4.23).

Theorem 5.30: EQ_{TM} is not Turing-recognizable

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs with } L(M_1) = L(M_2) \}$$
$$\subseteq \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs} \} \equiv \Omega_{EQ}$$

Proof. Reduce $\overline{A_{TM}} \leq_m EQ_{TM}$, and apply Corollary 5.29.

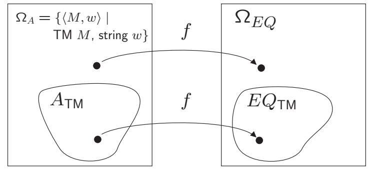


- Reduction $f(\langle M, w \rangle) = \langle M_1, M_2 \rangle$
  - ▲ $M_1 =$ "*reject* on all inputs."
  - ▲ $M_2 =$ "On input x:
    1. Ignore input x, and run M on w.
    2. If M accepts w, *accept*; if M rejects w, *reject*."
- $L(M_1) = \emptyset$.
- If M accepts w (i.e., $\langle M, w \rangle \notin \overline{A_{TM}}$), then $L(M_2) = \Sigma^*$.
  If M doesn't accept w (i.e., $\langle M, w \rangle \in \overline{A_{TM}}$), then $L(M_2) = \emptyset$.
- Thus, $\langle M, w \rangle \in \overline{A_{TM}} \iff f(\langle M, w \rangle) = \langle M_1, M_2 \rangle \in EQ_{TM}$.
- $\overline{A_{TM}}$ not TM-recognizable (Cor. 4.23),
  so EQ_{TM} not TM-recognizable by Corollary 5.29.

Theorem 5.30: EQ_{TM} is not co-Turing-recognizable

$$EQ_{\mathsf{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs with } L(M_1) = L(M_2) \}$$
$$\subseteq \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs} \} \equiv \Omega_{EQ}$$

Proof. Reduce $A_{\mathsf{TM}} \leq_{\mathsf{m}} EQ_{\mathsf{TM}}$, and apply Fact on slide 5-34.



- Reduction $f(\langle M, w \rangle) = \langle M_1, M_2 \rangle$
  - ▲ M_1 = "*accept* on all inputs."
  - ▲ M_2 = "On input x:
    1. Ignore input x, and run M on w.
    2. If M accepts w, *accept*; if M rejects w, *reject*."
- $L(M_1) = \Sigma^*$.
- If M accepts w (i.e., $\langle M, w \rangle \in A_{\mathsf{TM}}$), then $L(M_2) = \Sigma^*$.
  If M doesn't accept w (i.e., $\langle M, w \rangle \notin A_{\mathsf{TM}}$), then $L(M_2) = \emptyset$.
- $\langle M, w \rangle \in A_{\mathsf{TM}} \iff f(\langle M, w \rangle) = \langle M_1, M_2 \rangle \in EQ_{\mathsf{TM}}$.
- Because A_{TM} is not co-Turing-recognizable,
  EQ_{TM} is not co-Turing-recognizable by Fact on slide 5-34.

Summary of Chapter 5

- Computable function $f : \Sigma_1^* \to \Sigma_2^*$ has TM that maps
  - strings in Σ_1^* (i.e., instances of one problem)
  - to strings in Σ_2^* (i.e., instances of another problem)
- Mapping reduction $A \leq_{\mathsf{m}} B$:
  $w \in A \iff f(w) \in B$, for some computable function f.
  - If I can solve B, then I can solve A.
  - If I can't solve A, then I can't solve B.
- Undecidable problems: A_{TM}, $HALT_{\mathsf{TM}}$, E_{TM}, REG_{TM}, EQ_{TM}, ALL_{CFG}
- Rice's Theorem: any nontrivial property of the language of a TM is undecidable.
- E_{TM} is not Turing-recognizable.
- EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.