CS 341, Fall 2011
Solutions for Quiz 2, Day Section

1. (a) There are two different approaches one can use to show that $\mathit{CLIQUE} \in \mathrm{NP}$:

   - Show that *CLIQUE* has a nondeterministic Turing machine that runs in polynomial time, or
   - Show that *CLIQUE* has a polynomial-time verifier.

   We will use the second approach.

   To do this, let the certificate c for the verifier be a list of the nodes in the graph G forming a k-clique. Below is a verifier for *CLIQUE*:

   $V =$ "On input $\langle\langle G, k\rangle, c\rangle$:

      1. Test whether c is a set of k different nodes in G.
      2. Test whether G contains all edges connecting nodes in c.
      3. If both tests pass, *accept*; otherwise, *reject*."

   We now have to show V runs in polynomial time; i.e., if the graph G has m nodes and a k-clique, the time V needs to verify $\langle G, k\rangle \in \mathit{CLIQUE}$ is a polynomial function of m and k. Stage 1 of V requires checking if c consists of k different nodes from G, and this takes $O(k)O(n) = O(kn)$ time. Stage 2 needs to check G has an edge connecting each pair of nodes in c; since there are $O(k^2)$ pairs and $O(m^2)$ edges in G, this stage requires $O(k^2)O(m^2) = O(k^2m^2)$ time. Stage 3 takes constant time. Thus, the overall complexity of V is $O(k^2m^2)$, which is polynomial in m and k, so V is a polynomial-time verifier.

   (b) We need to show that $\mathit{3SAT} \leq_{\mathrm{P}} \mathit{CLIQUE}$, where we will show in the next part that the complexity of the reduction is polynomial. Consider a 3cnf-function ϕ, and we will show how to convert ϕ into a graph G such that $\phi \in \mathit{3SAT}$ if and only if $G \in \mathit{CLIQUE}$. To construct G from ϕ, suppose that ϕ has k clauses, which each consist of 3 literals. Then do the following:

   - For each clause i in ϕ, define a triple t_i of 3 nodes for G, with one node for each literal in clause i.
   - Add an edge between each pair of nodes, except
     - within the same triple
     - between contradictory literals, e.g., x and $\overline{x}$.

   We now show that $\phi \in \mathit{3SAT}$ iff $\langle G, k\rangle \in \mathit{CLIQUE}$. To show that $\phi \in \mathit{3SAT}$ implies $G \in \mathit{CLIQUE}$, suppose that ϕ has m variables and k clauses, and suppose that $(x_1, x_2, \ldots, x_m) \in \{0,1\}^m$ satisfies ϕ. This assignment thus ensures that each clause has (at least) one literal that evaluates to true. Then choose the nodes in G that correspond to those k literals. This set of k nodes are from k different triples. Moreover, the k literals don't contradict each other since ϕ is satisfied. Thus, the k nodes form a k-clique since G included all edges except between pairs of nodes in the same triple and between contradictory pairs of nodes. Hence, $\langle G, k\rangle \in \mathit{CLIQUE}$.

We now show that $\langle G, k\rangle \in \mathit{CLIQUE}$ implies $\phi \in \mathit{3SAT}$. Consider the k nodes in G forming the k-clique. Since G includes all edges except between pairs of nodes in the same triple and between contradictory pairs of nodes, the k-clique must contain k nodes from k different triples and the literals corresponding to those nodes are not contradicting. Thus, in each clause i of ϕ, set the literal corresponding to the chosen node in the k-clique to 1. This then gives a satisfying assignment for ϕ, so $\phi \in \mathit{3SAT}$.

(c) We now show that converting the 3cnf-function ϕ into the graph G takes polynomial time. Suppose that ϕ has m variables and k clauses. We need to show that the size of G is polynomial in k and m. Since G has a triple of nodes for each clause, G then has $3k$ nodes. Edges connect every pair of nodes except nodes in the same triple and contradictory literals. Thus, the number of edges in G is less than $\binom{3k}{2} = O((3k)^2) = O(k^2)$, which is polynomial in the size of ϕ, so the size of G is polynomial in k.