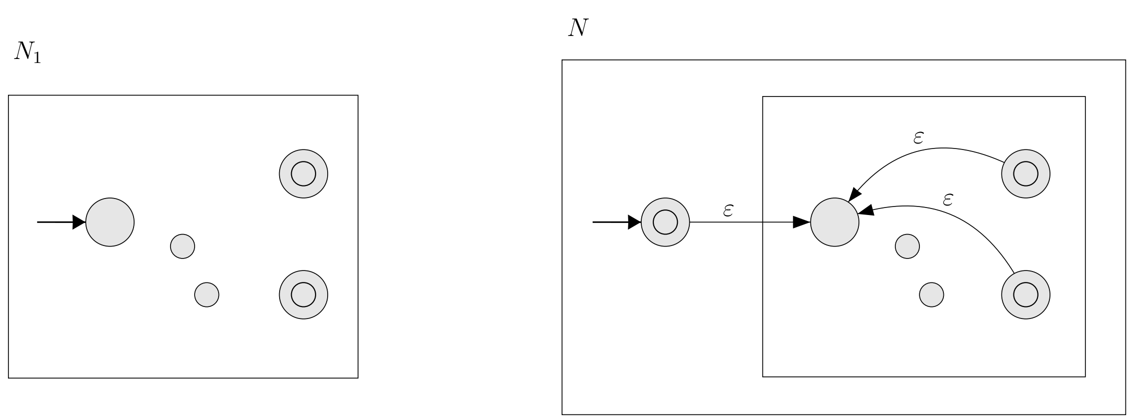
CS 341, Spring 2013, Face-to-Face Section
Solutions for Midterm 1

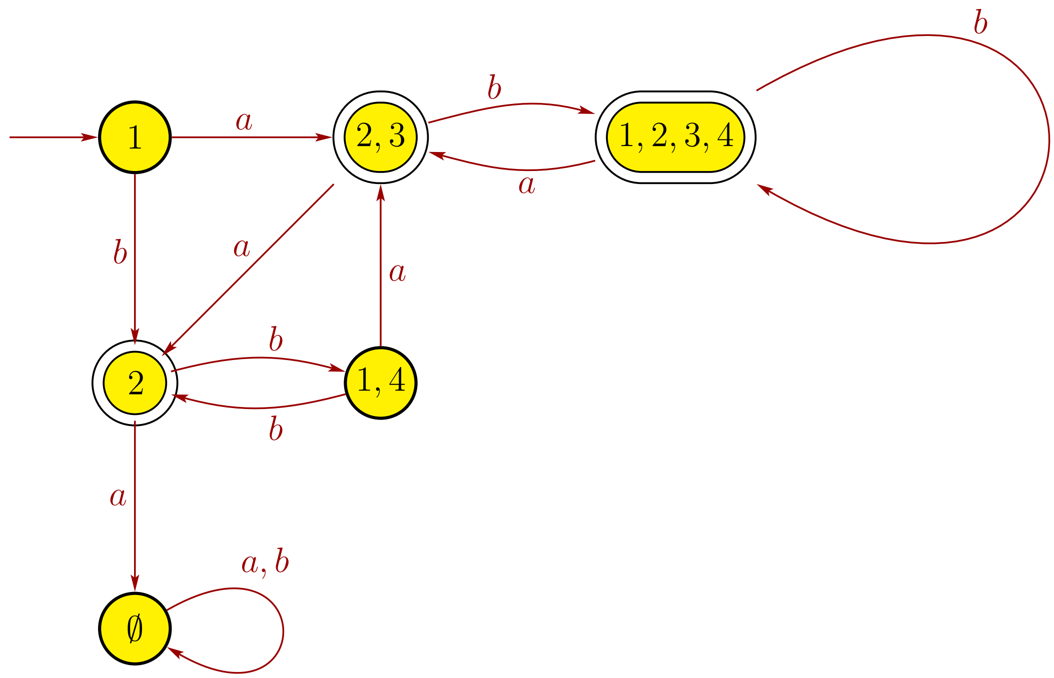
1. (a) False. Let $A = \{ a^n b^n \mid n \geq 0 \}$ and $B = (a \cup b)^*$. Then $A \subseteq B$, A is nonregular, and B is regular.
   (b) False. Let $A = \emptyset$ and $B = \{ a^n b^n \mid n \geq 0 \}$. Then $A \subseteq B$, A is regular since it's finite, and B is nonregular.
   (c) False. The language a^* is regular but infinite.
   (d) False. $A = \{a^n b^n \mid n \geq 0\}$ is context-free but not regular.
   (e) True. Homework 2, problem 5.
   (f) False. 0^*1^* generate the string $001 \notin A$, so the regular expression is not correct. In fact, A is nonregular, so it can't have a regular expression.
   (g) False. If A has an NFA, then Corollary 1.40 implies that A is regular.
   (h) True. Corollary 2.32.
   (i) True, by Lemma 2.27 and Theorem 2.9.
   (j) False. The transition function of an NFA is $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$.

2. (a) $(\varepsilon \cup 1)(01)^*00(10)^*(\varepsilon \cup 1) \;\cup\; (\varepsilon \cup 0)(10)^*11(01)^*(\varepsilon \cup 0)$
   (b) $(aa \cup b)a^*bb^*$ or $(aaa^* \cup ba^*)bb^*$ or ...
   (c) As on slide 1-53, an NFA N for A_1^* is as below:



   (d) (Homework 5, problem 3b.) A CFG for $A_1 \circ A_2$ is $G_3 = (V_3, \Sigma, R_3, S_3)$ with $V_3 = V_1 \cup V_2 \cup \{S_3\}$ and $R_3 = R_1 \cup R_2 \cup \{ S_3 \to S_1 S_2 \}$.

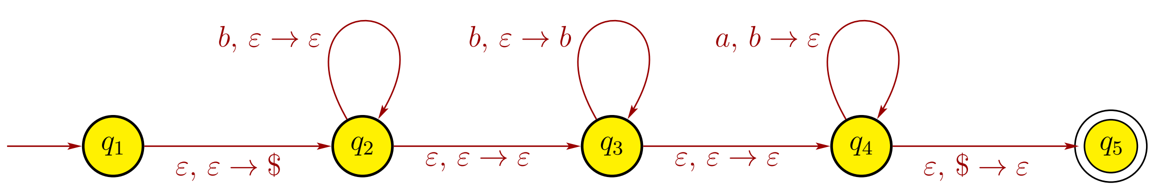
3. A DFA for C is below:



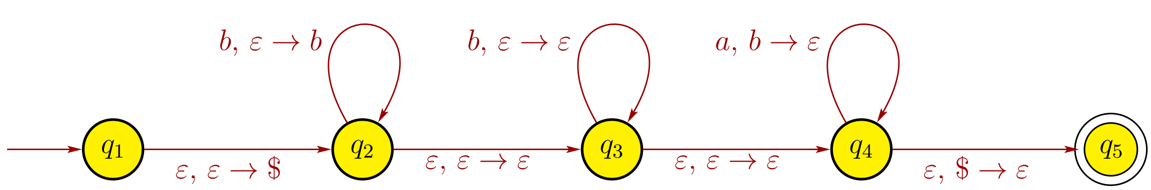
4. (a) $G = (V, \Sigma, R, S)$ with set of variables $V = \{S, Z\}$, where S is the start variable; set of terminals $\Sigma = \{a, b\}$; and rules

$$\begin{aligned} S &\rightarrow bSa \mid Z \\ Z &\rightarrow bZ \mid \varepsilon \end{aligned}$$

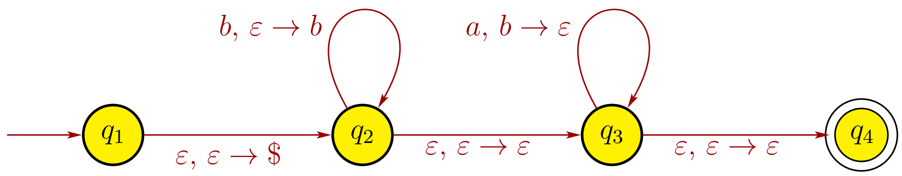
(b) There are infinitely many correct PDAs for L. The below PDA guesses how many b's not to match to the a's (state q_2), then pushes the b's to match with the a's (state q_3), matches the a's with the pushed b's (state q_4), and finally checks that the stack is empty (transition from q_4 to q_5).



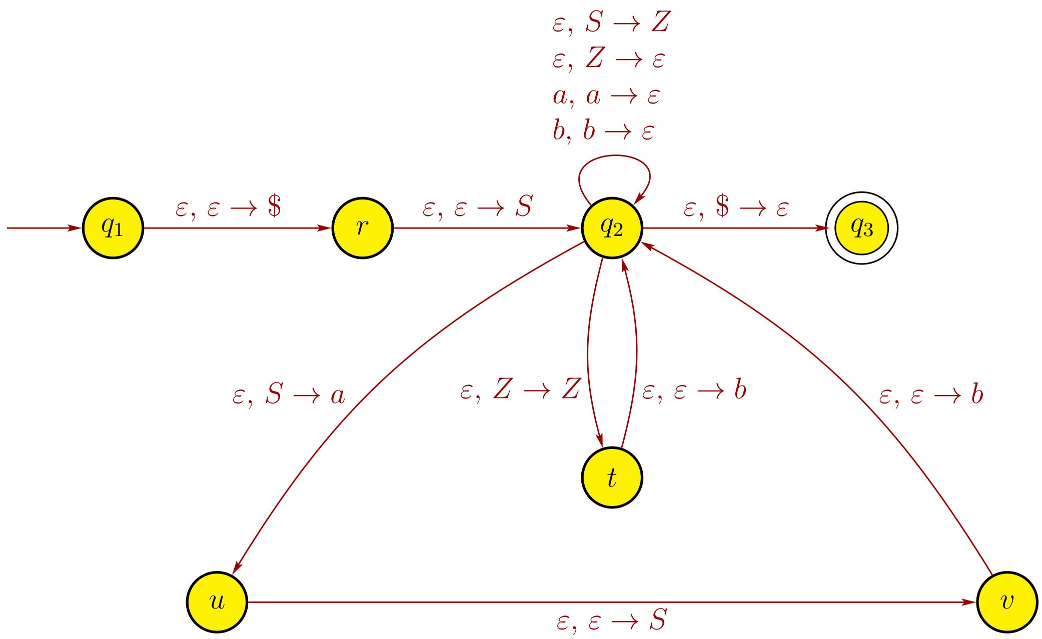
Below is another PDA for L, which first pushes b's to match the a's (state q_2), then guesses how many b's not to match with a's (state q_3), matches the a's with the pushed b's (state q_4), and finally checks that the stack is empty (transition from q_4 to q_5).



Below is yet another PDA for L. This one pushes all of the b's onto the stack (state q_2), and matches the a's with some of the pushed b's (state q_3). This PDA can accept a string with symbols (b's and \$) still on the stack.



Yet another approach uses the algorithm from Lemma 2.21 to convert the CFG in part (a) into a PDA.



5. Language A is nonregular. We prove this by contradiction. Suppose that A is a regular language. Let p be the "pumping length" of the Pumping Lemma. Consider the string $s = b^p a^p$. Note that $s \in A$, and $|s| = 2p > p$, so the Pumping Lemma will hold. Thus, there exists strings x, y, and z such that $s = xyz$ and

   (a) $xy^iz \in A$ for each $i \geq 0$,

   (b) $|y| > 0$,

   (c) $|xy| \leq p$.

   Since the first p symbols of s are all b's, the third property implies that x and y consist only of b's. So z will be the rest of the b's, followed by a^p. The second property states that $|y| > 0$, so y has at least one b. More precisely, we can then say that

$$x = b^j \text{ for some } j \geq 0,$$

$$\begin{aligned} y &= b^k \text{ for some } k \geq 1, \\ z &= a^p \text{ for some } m \geq 0. \end{aligned}$$

Since $b^p a^p = s = xyz = b^j b^k b^m a^p = b^{j+k+m} a^p$, we must have that

$$j + k + m = p \quad \text{ and } \quad k \geq 1.$$

The first property implies that $xy^0 z = xz \in A$, but

$$\begin{aligned} xz &= b^j b^m a^p \\ &= b^{j+m} a^p \notin A \end{aligned}$$

since $j + m < p$ because $j + k + m = p$ and $k \geq 1$, so the number of b's in s is less than the number of a's. This is a contradiction. Therefore, A is a nonregular language.