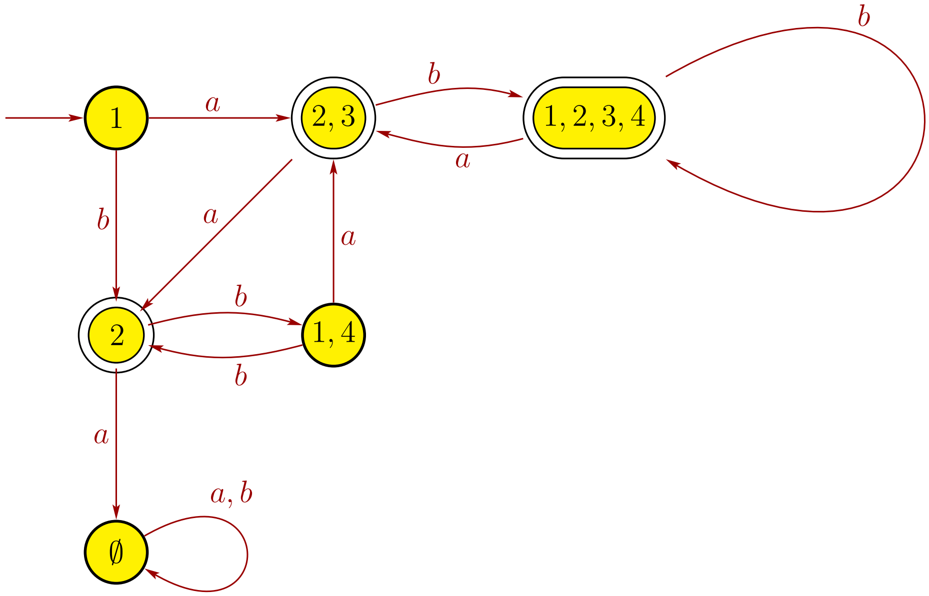
# CS 341, Fall 2016, Face-to-Face Section
# Solutions for Midterm 1

1. (a) False. $A = \{a^n b^n \mid n \geq 0\}$ is context-free but not regular.
   (b) True. Homework 2, problem 5.
   (c) False. $0^*1^*$ generate the string $001 \notin A$, so the regular expression is not correct. In fact, $A$ is nonregular, so it can't have a regular expression.
   (d) False. If $A$ has an NFA, then Corollary 1.40 implies that $A$ is regular.
   (e) True. Corollary 2.32.
   (f) True, by Lemma 2.27 and Theorem 2.9.
   (g) False. The transition function of an NFA is $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$.
   (h) False. Let $A = \{ a^n b^n \mid n \geq 0 \}$ and $B = (a \cup b)^*$. Then $A \subseteq B$, $A$ is nonregular, and $B$ is regular.
   (i) False. Let $A = \emptyset$ and $B = \{ a^n b^n \mid n \geq 0 \}$. Then $A \subseteq B$, $A$ is regular since it's finite, and $B$ is nonregular.
   (j) False. The language $a^*$ is regular but infinite.

2. (a) $b^*ab^* \cup b^*aa^*bb^*$. Another regular expression is $b^*(a \cup aa^*b)b^*$. There are infinitely many regular expressions for the language.
   (b) $G' = (V', \Sigma, R', S_0)$, where $V' = V \cup \{S_0\}$, $S_0$ is the (new) starting variable, $\Sigma$ is the same alphabet of terminals as in $G$, and $R' = R \cup \{S_0 \to SS_0 \mid \varepsilon\}$.
   (c) $M_3 = (Q_3, \Sigma, \delta_3, q_3, F_3)$, where $Q_3 = Q_1 \times Q_2$; $\Sigma$ is the same alphabet as $M_1$ and $M_2$ have; the transition function $\delta_3$ satisfies $\delta_3((q,r),\ell) = (\delta_1(q,\ell), \delta_2(r,\ell))$ for $(q,r) \in Q_3$ and $\ell \in \Sigma$; the starting state $q_3 = (q_1, q_2)$; and $F_3 = (Q_1 \times F_2) \cap (F_1 \times Q_2)$, which also can be written as $F_1 \times F_2$.
   (d) After one step, the CFG is then

$$\begin{aligned} S_0 &\to S \\ S &\to A1SA \mid 1SA \mid A1S \mid 1S \mid A0 \mid 0 \mid \varepsilon \\ A &\to 0S0 \end{aligned}$$

3. A DFA for $C$ is below:

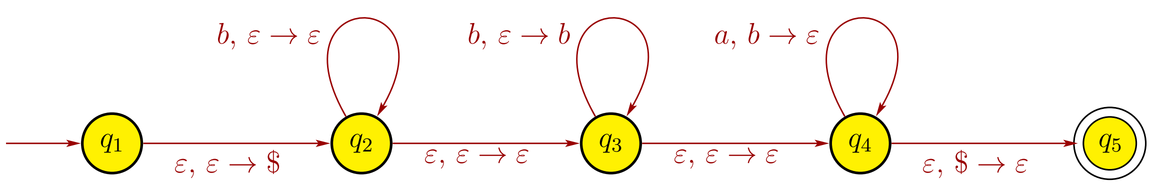



4. (a) $G = (V, \Sigma, R, S)$ with set of variables $V = \{S, Z\}$, where $S$ is the start variable; set of terminals $\Sigma = \{a, b\}$; and rules

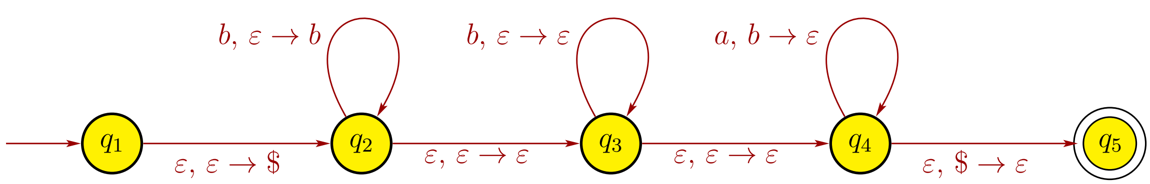
$$\begin{aligned} S &\rightarrow bSa \mid Z \\ Z &\rightarrow bZ \mid \varepsilon \end{aligned}$$

There are infinitely many other correct CFGs for $L$.

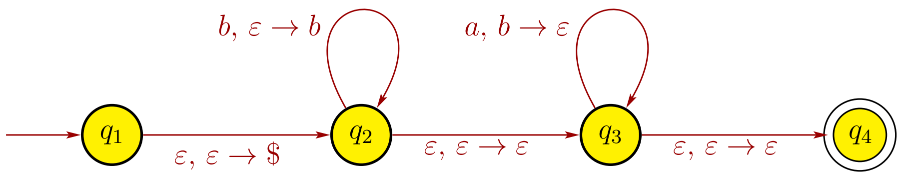
(b) There are infinitely many correct PDAs for $L$. The below PDA guesses how many $b$'s not to match to the $a$'s (state $q_2$), then pushes the $b$'s to match with the $a$'s (state $q_3$), matches the $a$'s with the pushed $b$'s (state $q_4$), and finally checks that the stack is empty (transition from $q_4$ to $q_5$).



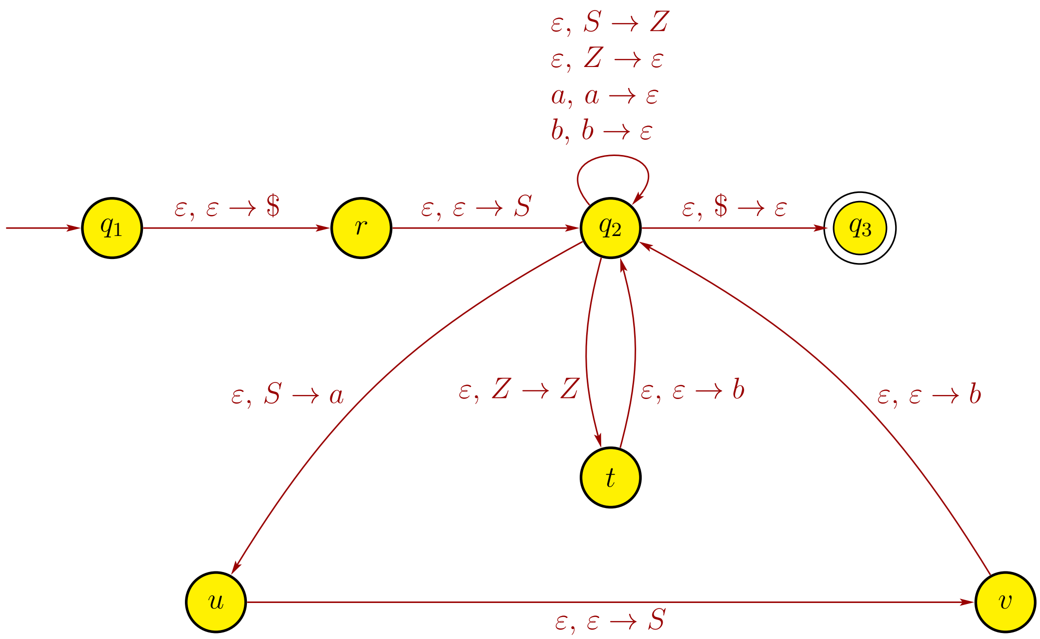

Below is another PDA for $L$, which first pushes $b$'s to match the $a$'s (state $q_2$), then guesses how many $b$'s not to match with $a$'s (state $q_3$), matches the $a$'s with the pushed $b$'s (state $q_4$), and finally checks that the stack is empty (transition from $q_4$ to $q_5$).



Below is yet another PDA for $L$. This one pushes all of the $b$'s onto the stack (state $q_2$), and matches the $a$'s with some of the pushed $b$'s (state $q_3$). This PDA can accept a string with symbols ($b$'s and \$) still on the stack.




Yet another approach uses the algorithm from Lemma 2.21 to convert the CFG in part (a) into a PDA.




5. Language $A$ is nonregular. We prove this by contradiction. Suppose that $A$ is a regular language. Let $p$ be the "pumping length" of the Pumping Lemma. Consider the string $s = b^p a^p$. Note that $s \in A$, and $|s| = 2p > p$, so the Pumping Lemma will hold. Thus, there exists strings $x$, $y$, and $z$ such that $s = xyz$ and

   (a) $xy^i z \in A$ for each $i \geq 0$,

   (b) $|y| > 0$,

   (c) $|xy| \leq p$.

   Since the first $p$ symbols of $s$ are all $b$'s, the third property implies that $x$ and $y$ consist only of $b$'s. So $z$ will be the rest of the $b$'s, followed by $a^p$. The second property states that $|y| > 0$, so $y$ has at least one $b$. More precisely, we can then say that

$$x = b^j \text{ for some } j \geq 0,$$

$$\begin{aligned} y &= b^k \text{ for some } k \geq 1, \\ z &= b^m a^p \text{ for some } m \geq 0. \end{aligned}$$

Since $b^p a^p = s = xyz = b^j b^k b^m a^p = b^{j+k+m} a^p$, we must have that

$$j + k + m = p \quad \text{and} \quad k \geq 1.$$

The first property implies that $xy^0 z = xz \in A$, but

$$\begin{aligned} xz &= b^j b^m a^p \\ &= b^{j+m} a^p \notin A \end{aligned}$$

since $j + m < p$ because $j + k + m = p$ and $k \geq 1$, so the number of $b$'s in $s$ is less than the number of $a$'s. This is a contradiction. Therefore, $A$ is a nonregular language.