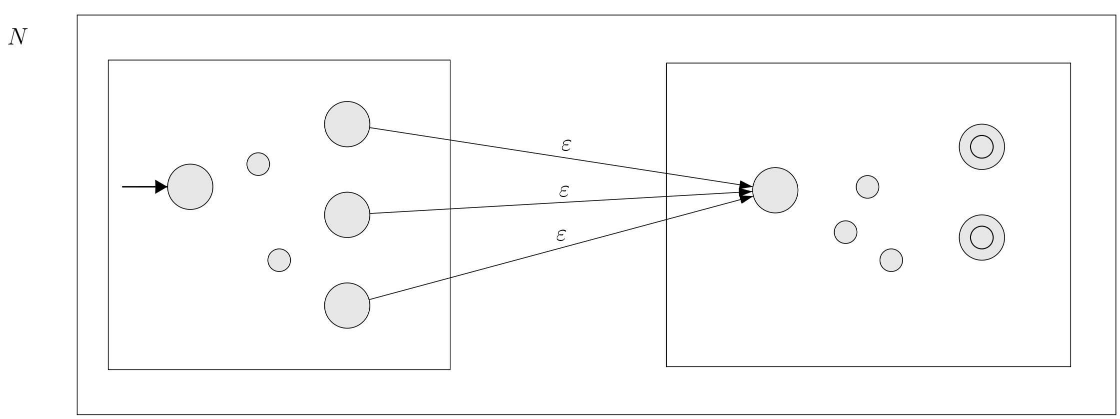
CS 341, Spring 2016
Solutions for Midterm, eLearning Section

1. (a) False. For example, let $A = \{a^n b^n \mid n \geq 0\}$ and $B = (a \cup b)^*$. Then $A \subseteq B$, B is regular, but A is nonregular.
   (b) False. a^*b^* generates the string $abb \notin \{a^n b^n \mid n \geq 0\}$. In fact, $\{\, a^n b^n \,|\, n \geq 0 \,\}$ is a nonregular language, so it cannot have a regular expression.
   (c) True. By Theorem 2.9. The fact that A is non-regular is irrelevant.
   (d) True. Because $\overline{A} \cap B \subseteq B$ and B is finite, we must have that $\overline{A} \cap B$ is finite. Thus, $\overline{A} \cap B$ is regular by slide 1-95. The fact that A has a PDA is irrelevant.
   (e) True. By Theorem 2.20.
   (f) False. For example, let $A = \{abc\}$ and $B = \{a^n b^n c^n \mid n \geq 0\}$, so $A \subseteq B$. Because A is finite, it is regular by slide 1-95. This implies A is also context-free by Corollary 2.32. But B is not context-free by slide 2-96.
   (g) False. The TM M can also loop on w.
   (h) False. $A = \{a^n b^n c^n \mid n \geq 0\}$ is nonregular and not context-free.
   (i) False. The language a^* is regular but infinite.
   (j) True. Suppose A is nonregular and finite. But each finite language is regular by slide 1-95, which is a contradiction.

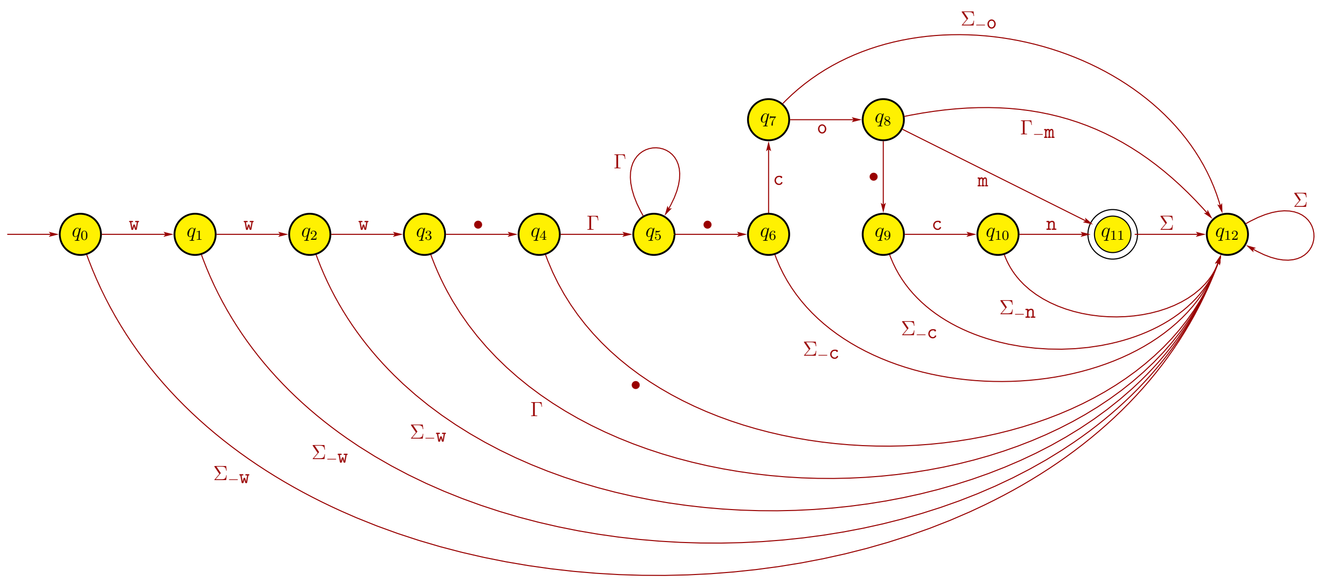
2. (a) $\varepsilon \cup a \cup b \cup (a \cup b)^*(ab \cup ba)$. There are other correct regular expressions.
   (b)
   - $X \to SY$ is improper since the start variable S can't be on the right side.
   - $X \to \varepsilon$ is improper if ε is on the right side, S must be on the left side.
   - $Y \to Xa$ is improper since the right side has a mix of terminals and variables.
   - $Y \to ab$ is improper since a rule can't have more than one terminal on the right side.
   - $Y \to X$ is improper since it is a unit rule.

   (c) As given on slide 1-63, $A_1 \circ A_2$ has the following NFA N:



(d) This is Homework 5, problem 3a. The language $A_3 = A_1 \cup A_2$ has a CFG $G_3 = (V_3, \Sigma, R_3, S_3)$, with $V_3 = V_1 \cup V_2 \cup \{S_3\}$ and $R_3 = R_1 \cup R_2 \cup \{\, S_3 \rightarrow S_1,\ S_3 \rightarrow S_2 \,\}$, where S_3 is the start variable.

3. Below is a DFA for the language L. There are other correct DFAs for L.



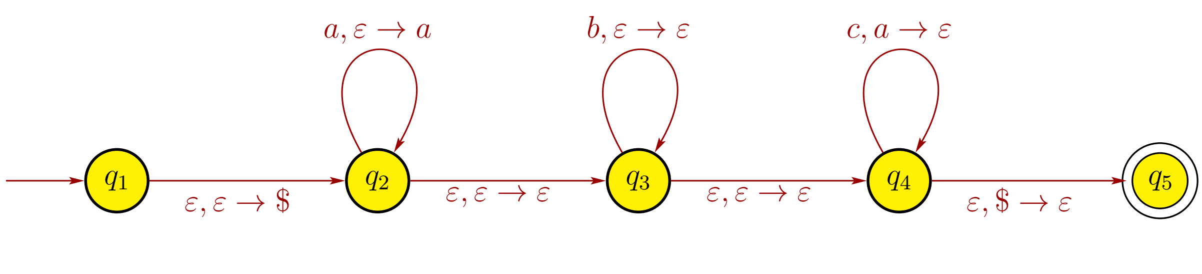
4. $q_1 0\#0 \quad xq_2\#0 \quad x\#q_4 0 \quad xq_6\#x \quad q_7 x\#x \quad xq_1\#x \quad x\#q_8 x \quad x\#xq_8$
$x\#x\sqcup q_{\text{accept}}$

5. (a) CFG $G = (V, \Sigma, R, S)$, with $V = \{S, X\}$ and start variable S, $\Sigma = \{a, b, c\}$, and rules R:

$$S \rightarrow aSc \mid X$$
$$X \rightarrow bX \mid \varepsilon$$

There are other correct CFGs.

(b) PDA



There are other correct PDAs.

6. This is HW 6, problem 2b. We will use a proof by contradiction, so we first assume the opposite of what we want to show; i.e., suppose the following is true:

   R1. The class of context-free languages is closed under complementation.

   Let $A = \{a^n b^k c^n \mid n, k \geq 0\}$ and $B = \{a^n b^n c^k \mid n, k \geq 0\}$. In problem 5, we gave a CFG for A, so A is context-free. A CFG for B has rules

$$S \to XY$$
$$X \to aXb \mid \varepsilon$$
$$Y \to cY \mid \varepsilon$$

   so B is also context-free. Then R1 implies $\overline{A}$ and $\overline{B}$ are context-free. We know the class of context-free languages is closed under union, as shown on slide 2-101, so we then must have that $\overline{A} \cup \overline{B}$ is context-free. Then again apply R1 to conclude that $\overline{\overline{A} \cup \overline{B}}$ is context-free. Now DeMorgan's law states that $A \cap B = \overline{\overline{A} \cup \overline{B}}$. But $A \cap B = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free, as shown on slide 2-96. This is a contradiction, so R1 must not be true.

7. Suppose that A is a regular language. Let p be the pumping length, and consider the string $s = a^p c^p \in A$. Note that $|s| = 2p \geq p$, so the pumping lemma implies we can write $s = xyz$ with $xy^i z \in A$ for all $i \geq 0$, $|y| > 0$, and $|xy| \leq p$. Now, $|xy| \leq p$ implies that x and y have only a's (together up to p in total) and z has the rest of the a's at the beginning, followed by c^p. Hence, we can write $x = a^j$ for some $j \geq 0$, $y = a^k$ for some $k \geq 0$, and $z = a^\ell c^p$, where $j + k + \ell = p$ since $xyz = s = a^p c^p$. Also, $|y| > 0$ implies $k > 0$. Now consider the string $xyyz = a^j a^k a^k a^\ell c^p = a^{p+k} c^p$ since $j + k + \ell = p$. Note that $xyyz \notin A$ since $k > 0$ so the number of a's and c's are not equal. This contradicts (i) of the pumping lemma, so A is not a regular language.