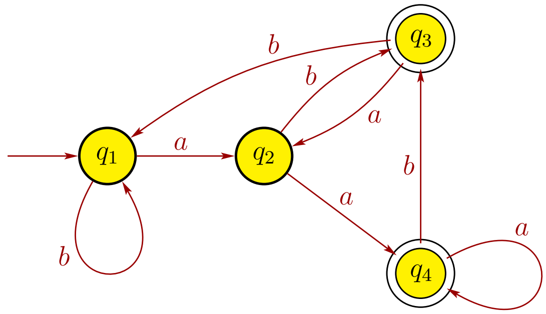
CS 341, Spring 2018
Solutions for Midterm, eLearning Section

1. (a) False. Let $A = \emptyset$ and $B = \{ a^n b^n \mid n \geq 0 \}$. Then $A \subseteq B$, A is regular because it's finite, but B is nonregular.
   (b) False. The language a^* is regular but infinite.
   (c) False. $A = \{a^n b^n \mid n \geq 0\}$ is context-free but not regular.
   (d) True. Homework 2, problem 5.
   (e) False. 0^*1^* generate the string $001 \notin A$, so the regular expression is not correct. In fact, A is nonregular, so it can't have a regular expression.
   (f) False. If A has an NFA, then Corollary 1.40 implies that A is regular.
   (g) True. Corollary 2.32 implies that A is a CFL, so it has a CFG. Then Theorem 2.20 ensures that A has a PDA.
   (h) True, by Lemma 2.27 and Theorem 2.9.
   (i) False. The transition function of an NFA is $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$.
   (j) False. Let $A = \{ a^n b^n \mid n \geq 0 \}$ and $B = (a \cup b)^*$. Then $A \subseteq B$, A is nonregular, and B is regular.

2. (a) This is HW 3, problem 4f. $(\varepsilon \cup b)(ab)^* aa(ba)^*(\varepsilon \cup b) \;\cup\; (\varepsilon \cup a)(ba)^* bb(ab)^*(\varepsilon \cup a)$ is a regular expression for language A. There are infinitely many other correct regular expressions for A.
   (b) After one step, the CFG is then

$$
\begin{aligned}
S_0 &\to S \\
S &\to A1SA \mid 1SA \mid A1S \mid 1S \mid A0 \mid 0 \mid \varepsilon \\
A &\to 0S0
\end{aligned}
$$

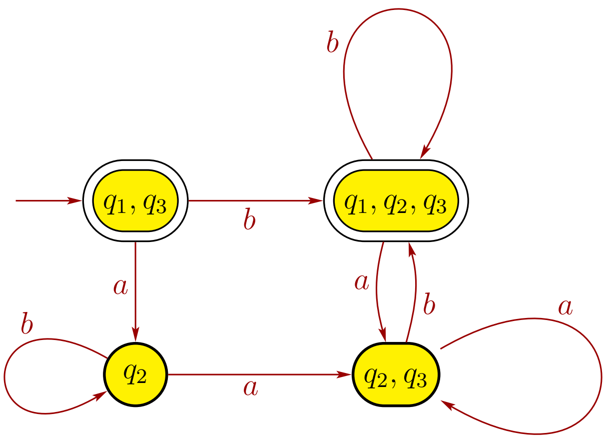
   (c) This is basically Homework 2, problem 2(g). A DFA for C is



   There are infinitely many other correct DFAs for C.
   (d) This is Homework 5, problem 3b. A CFG for $A_1 \circ A_2$ is $G_3 = (V_3, \Sigma, R_3, S_3)$ with $V_3 = V_1 \cup V_2 \cup \{S_3\}$, where $S_3 \notin V_1 \cup V_2$, S_3 is the starting variable of G_3, and $R_3 = R_1 \cup R_2 \cup \{ S_3 \to S_1 S_2 \}$.

3. $q_1 0\#0 \quad x q_2 \#0 \quad x\#q_4 0 \quad x q_6 \#x \quad q_7 x\#x \quad x q_1 \#x \quad x\#q_8 x \quad x\#x q_8$
$x\#x \sqcup q_{\text{accept}}$

4. DFA

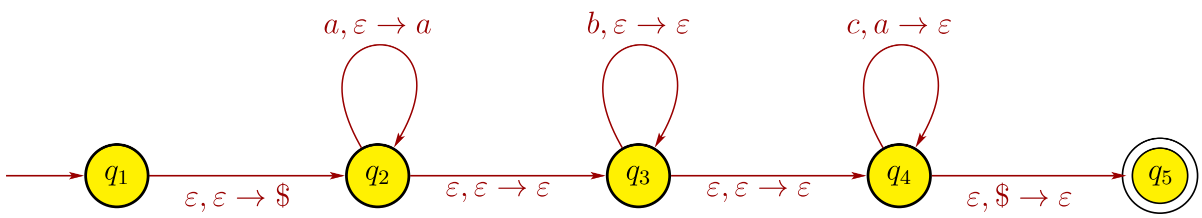


5. (a) CFG $G = (V, \Sigma, R, S)$, with $V = \{S, X\}$ and start variable S, $\Sigma = \{a, b, c\}$, and rules R:

$$S \to aSc \mid X$$
$$X \to bX \mid \varepsilon$$

There are other correct CFGs.

(b) PDA



State q_2 pushes an a for each a read. State q_3 reads all of the b's in the middle, but doesn't alter the stack. State q_4 pops an a for each c read. The transition from q_4 to q_5 makes sure the stack is empty.
There are other correct PDAs.

6. This is HW 6, problem 2b. We will use a proof by contradiction, so we first assume the opposite of what we want to show; i.e., suppose the following is true:

R1. The class of context-free languages is closed under complementation.

Let $A = \{a^n b^k c^n \mid n, k \geq 0\}$ and $B = \{a^n b^n c^k \mid n, k \geq 0\}$. In problem 5, we gave a CFG for A, so A is context-free. A CFG for B has rules

$$S \to XY$$
$$X \to aXb \mid \varepsilon$$
$$Y \to cY \mid \varepsilon$$

so B is also context-free. Then R1 implies $\overline{A}$ and $\overline{B}$ are context-free. We know the class of context-free languages is closed under union, as shown on slide 2-101, so we then must have that $\overline{A} \cup \overline{B}$ is context-free. Then again apply R1 to conclude that $\overline{\overline{A} \cup \overline{B}}$ is context-free. Now DeMorgan's law states that $A \cap B = \overline{\overline{A} \cup \overline{B}}$. But $A \cap B = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free, as shown on slide 2-96. This is a contradiction, so R1 must not be true.

7. Suppose that A is a regular language. Let p be the pumping length, and consider the string $s = a^p bba^p \in A$. Note that $|s| = 2p + 2 \geq p$, so the pumping lemma implies we can write $s = xyz$ with $xy^i z \in A$ for all $i \geq 0$, $|y| > 0$, and $|xy| \leq p$. Now, $|xy| \leq p$ implies that x and y have only a's (together up to p in total) and z has the rest of the a's at the beginning, followed by bba^p. Hence, we can write $x = a^j$ for some $j \geq 0$, $y = a^k$ for some $k \geq 0$, and $z = a^\ell bba^p$, where $j + k + \ell = p$ since $xyz = s = a^p bba^p$. Also, $|y| > 0$ implies $k > 0$. Now consider the string $xyyz = a^j a^k a^k a^\ell bba^p = a^{p+k} bba^p$ since $j + k + \ell = p$. Note that $xyyz \notin A$ since it is not the same forwards and backwards because $k > 0$, which contradicts (i), so A is not a regular language.