CS 341, Fall 2019, Hybrid Section
Solutions for Midterm 1

1. (a) False. For example, $A = \{0^n 1^n 0^n \mid n \geq 0\}$ is a subset of $B = L((0 \cup 1)^*)$, but A is non-context-free and B is context-free.

   (b) False. The language $\{\, a^n b^n \mid n \leq 30\} = \{\varepsilon, ab, a^2b^2, a^3b^3, \ldots, a^{30}b^{30}\}$ is finite. Thus, slide 1-95 implies the language is regular.

   (c) True. Because A has a regular expression, A is a regular language by Theorem 1.54. Then Corollary 2.32 implies A is also context-free, so it has a CFG. Theorem 2.9 then ensures that A has a CFG in Chomsky normal form.

   (d) True. See slide 2-111.

   (e) True. HW 4, problem 5(a).

   (f) True. HW 4, problem 5(c).

   (g) True. Suppose A is non-context-free but regular. But then Corollary 2.32 implies A is context-free, which is a contradiction.

   (h) False. The language with regular expression 1^* is regular by Kleene's Theorem (Theorem 1.54), but this language is infinite.

   (i) True. By HW 2, problem 3, we know that $\overline{A}$ is regular. Because $\overline{A}$ and B are regular, then $\overline{A} \cup B$ is regular by Theorem 1.25. Theorem 1.49 then implies $(\overline{A} \cup B)^*$ is regular.

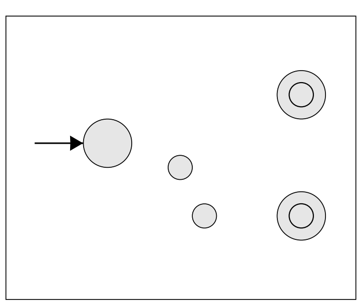
   (j) False. See HW 6, problem 2(a).

2. (a) $(\varepsilon \cup 1)(01)^*00(10)^*(\varepsilon \cup 1) \;\cup\; (\varepsilon \cup 0)(10)^*11(01)^*(\varepsilon \cup 0)$. There are infinitely many other correct regular expressions for this language.

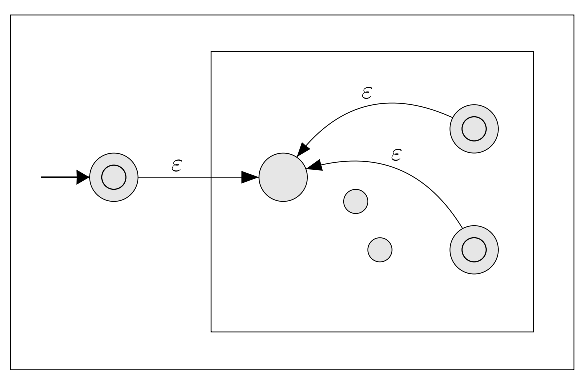
   (b) $(aa \cup b)a^*bb^*$. Another regular expression is $(aaa^* \cup ba^*)bb^*$. There are infinitely many correct regular expressions for this language.

   (c) As on slide 1-66 of the notes, if A_1 is defined by NFA N_1, then an NFA N for A_1^* is as below:

N_1

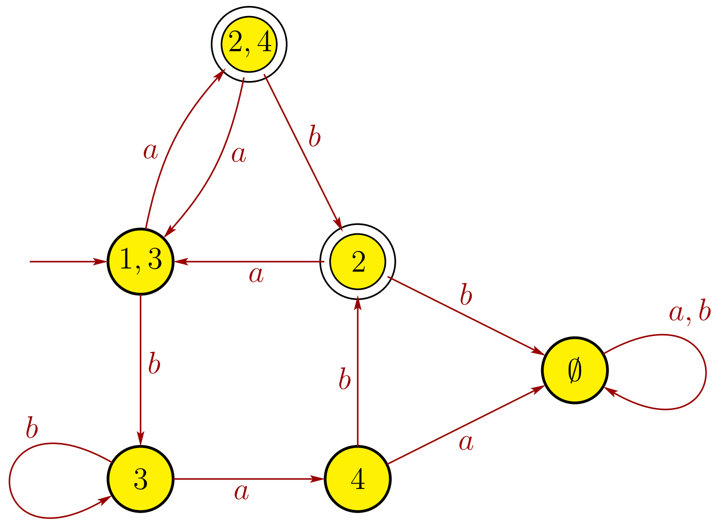


N



   (d) (Homework 5, problem 3b.) Assume that $S_3 \notin V_1 \cup V_2$. Then a CFG for $A_1 \circ A_2$ is $G_3 = (V_3, \Sigma, R_3, S_3)$ with $V_3 = V_1 \cup V_2 \cup \{S_3\}$ and $R_3 = R_1 \cup R_2 \cup \{\, S_3 \to S_1 S_2 \,\}$.

3. A DFA for C is below:

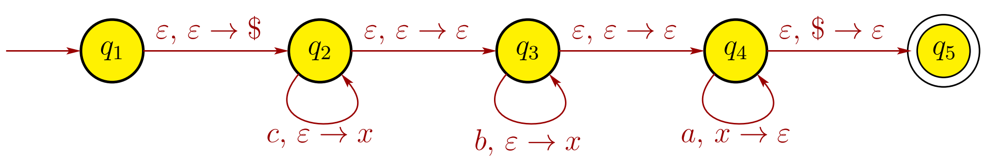


4. (a) $G = (V, \Sigma, R, S)$ with set of variables $V = \{S, X\}$, where S is the start variable; set of terminals $\Sigma = \{a, b, c\}$; and rules

$$\begin{aligned} S &\rightarrow cSa \mid X \\ X &\rightarrow bXa \mid \varepsilon \end{aligned}$$

There are infinitely many other correct CFGs for L.

(b) There are infinitely many correct PDAs for L. Here is one:



5. Language A is nonregular. We prove this by contradiction. Suppose that A is a regular language. Let p be the "pumping length" of the Pumping Lemma. Consider the string $s = a^p b a^p b a^p b$. Note that $s \in A$ because $s = www$ with $w = a^p b$. Also, we have that $|s| = 3p + 3 > p$, so the Pumping Lemma will hold. Thus, there exist strings x, y, and z such that $s = xyz$ and

   (a) $xy^i z \in A$ for each $i \geq 0$,

   (b) $|y| > 0$,

   (c) $|xy| \leq p$.

Because the first p symbols of s are all a's, the third property implies that x and y consist only of a's. So z will be the rest of the first set of a's (possibly none), followed by ba^pba^pb. The second property states that $|y| > 0$, so y has at least one a. More precisely, we can then say that

$$\begin{aligned} x &= a^j \text{ for some } j \geq 0, \\ y &= a^k \text{ for some } k \geq 1, \\ z &= a^m ba^p ba^p b \text{ for some } m \geq 0. \end{aligned}$$

Because

$$a^p ba^p ba^p b = s = xyz = a^j a^k a^m ba^p ba^p b = a^{j+k+m} ba^p ba^p b,$$

we must have that

$$j + k + m = p \quad \text{ and } \quad k \geq 1.$$

The first property implies that the pumped string $xy^2z \in A$, but

$$\begin{aligned} xy^2z &= a^j a^k a^k a^m ba^p ba^p b \\ &= a^{p+k} ba^p ba^p b \notin A. \end{aligned}$$

To see why $a^{p+k}ba^pba^pb \notin A$, note that when we split the original string $s = a^pba^pba^pb$ into equal thirds, each third was exactly the same, i.e., a^pb. But if we split the pumped string $a^{p+k}ba^pba^pb$ into equal thirds, the splitting locations shift to the left because $k > 0$, so the first third has only a's. But there are b's in the at least one of the other thirds, so we see that the pumped string $a^{p+k}ba^pba^pb$ cannot be written as www for some $w \in \Sigma^*$, i.e., $a^{p+k}ba^pba^pb \notin A$, which contradicts the first property of the pumping lemma. Therefore, A is a nonregular language.

Note that if you instead chose the string $s = a^pa^pa^p = a^{3p}$, you would not get a contradiction. This is because you could then choose x, y, z with $y = a^3$, and for any $i \geq 0$, the pumped string is

$$xy^iz = a^{3p+3(i-1)} = a^{3(p+i-1)} = a^{p+i-1}a^{p+i-1}a^{p+i-i} \in A,$$

so the first property of the pumping lemma holds, and there is no contradiction.