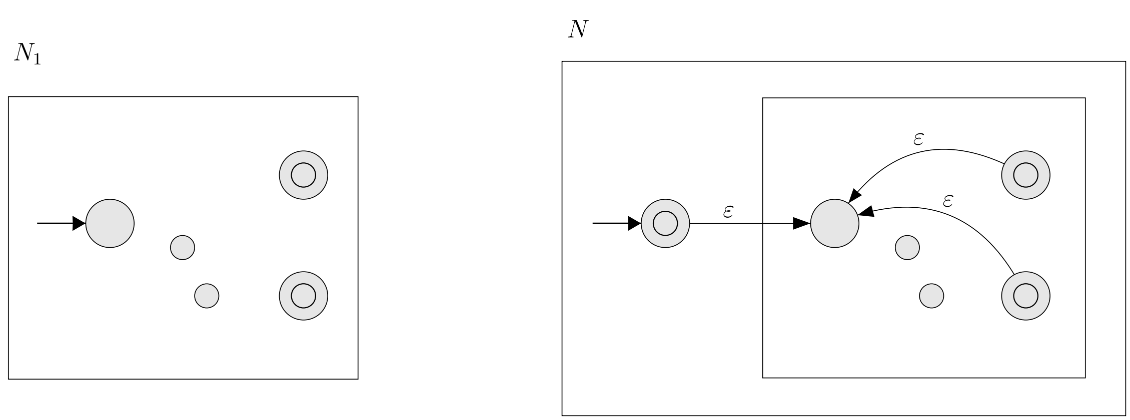
**CS 341, Fall 2021, Hybrid Section**
**Solutions for Midterm 1**

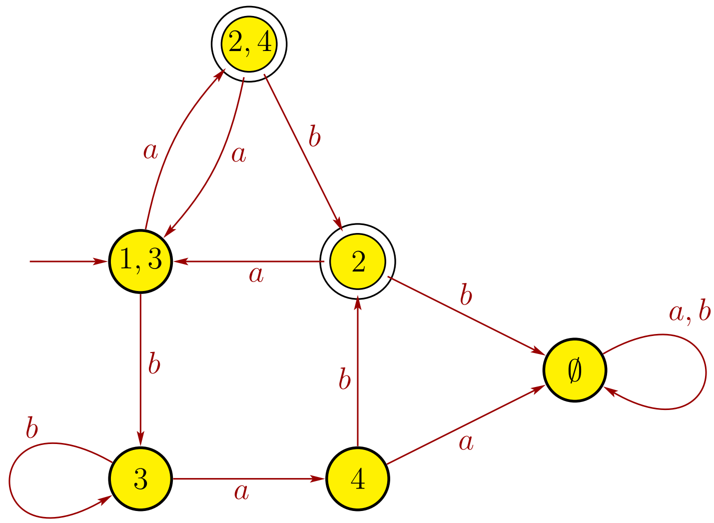
1. (a) False. Homework 6, problem 2(c).
   (b) True. HW 4, problem 5(a).
   (c) True. HW 4, problem 5(c).
   (d) True. Suppose $A$ is non-context-free but regular. But then Corollary 2.32 implies $A$ is context-free, which is a contradiction.
   (e) False. The language with regular expression $1^*$ is regular by Kleene's Theorem (Theorem 1.54), but this language is infinite.
   (f) True. By HW 2, problem 3, we know that $\overline{A}$ is regular. Because $\overline{A}$ and $B$ are regular, then $\overline{A} \cup B$ is regular by Theorem 1.25. Theorem 1.49 then implies $(\overline{A} \cup B)^*$ is regular.
   (g) False. See HW 6, problem 2(a).
   (h) False. For example, $A = \{0^n 1^n 0^n \mid n \geq 0\}$ is a subset of $B = L((0 \cup 1)^*)$, but $A$ is non-context-free and $B$ is context-free.
   (i) False. The language $\{ a^n b^n \mid n \leq 30\} = \{\varepsilon, ab, a^2b^2, a^3b^3, \ldots, a^{30}b^{30}\}$ is finite. Thus, slide 1-95 implies the language is regular.
   (j) True. Because $A$ has a regular expression, $A$ is a regular language by Theorem 1.54. Then Corollary 2.32 implies $A$ is also context-free, so it has a CFG. Theorem 2.9 then ensures that $A$ has a CFG in Chomsky normal form.

2. (a) $0^*1(10^*1 \cup 0)^*$
   There are infinitely many other correct regular expressions for this language.
   (b) $(aa \cup b)a^*ba^*$. Another regular expression is $(aaa^* \cup ba^*)ba^*$. There are infinitely many correct regular expressions for this language.
   (c) As on slide 1-66 of the notes, if $A_1$ is defined by NFA $N_1$, then an NFA $N$ for $A_1^*$ is as below:




   (d) (Homework 5, problem 3b.) Assume that $S_3 \notin V_1 \cup V_2$. Then a CFG for $A_1 \circ A_2$ is $G_3 = (V_3, \Sigma, R_3, S_3)$ with $V_3 = V_1 \cup V_2 \cup \{S_3\}$ and $R_3 = R_1 \cup R_2 \cup \{ S_3 \to S_1 S_2 \}$.

3. A DFA for $C$ is below:

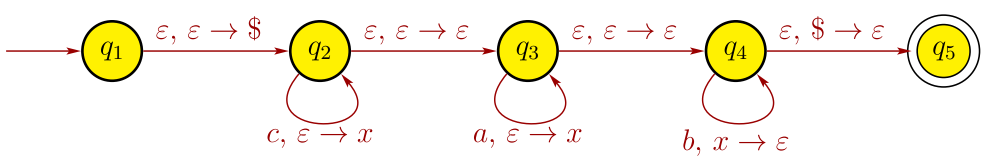



4. (a) $G = (V, \Sigma, R, S)$ with set of variables $V = \{S, X\}$, where $S$ is the start variable; set of terminals $\Sigma = \{a, b, c\}$; and rules

$$\begin{aligned} S &\rightarrow cSb \mid X \\ X &\rightarrow aXb \mid \varepsilon \end{aligned}$$

There are infinitely many other correct CFGs for $L$.

(b) There are infinitely many correct PDAs for $L$. Here is one:




In the above PDA, state $q_2$ pushes an $x$ for each $c$ read, state $q_3$ pushes an $x$ for each $a$ read, and state $q_4$ pops an $x$ for each $b$ read to match the $c$'s and $a$'s.

5. Language $A$ is nonregular. We prove this by contradiction. Suppose that $A$ is a regular language. Let $p$ be the "pumping length" of the Pumping Lemma. Consider the string $s = a^p b a^p b a^p b$. Note that $s \in A$ because $s = www$ with $w = a^p b$. Also, we have that $|s| = 3p + 3 > p$, so the Pumping Lemma will hold. Thus, there exist strings $x$, $y$, and $z$ such that $s = xyz$ and

(a) $xy^i z \in A$ for each $i \geq 0$,

(b) $|y| > 0$,

(c) $|xy| \le p$.

Because the first $p$ symbols of $s$ are all $a$'s, the third property implies that $x$ and $y$ consist only of $a$'s. So $z$ will be the rest of the first set of $a$'s (possibly none), followed by $ba^pba^pb$. The second property states that $|y| > 0$, so $y$ has at least one $a$. More precisely, we can then say that

$$\begin{aligned} x &= a^j \text{ for some } j \ge 0, \\ y &= a^k \text{ for some } k \ge 1, \\ z &= a^m ba^p ba^p b \text{ for some } m \ge 0. \end{aligned}$$

Because

$$a^p ba^p ba^p b = s = xyz = a^j a^k a^m ba^p ba^p b = a^{j+k+m} ba^p ba^p b,$$

we must have that

$$j + k + m = p \quad \text{and} \quad k \ge 1.$$

The first property implies that the pumped string $xy^2z \in A$, but

$$\begin{aligned} xy^2z &= a^j a^k a^k a^m ba^p ba^p b \\ &= a^{p+k} ba^p ba^p b \notin A. \end{aligned}$$

To see why $a^{p+k}ba^pba^pb \notin A$, note that when we split the original string $s = a^pba^pba^pb$ into equal thirds, each third was exactly the same, i.e., $a^pb$. But if we split the pumped string $a^{p+k}ba^pba^pb$ into equal thirds, the splitting locations shift to the left because $k > 0$, so the first third has only $a$'s. But there are $b$'s in the at least one of the other thirds, so we see that the pumped string $a^{p+k}ba^pba^pb$ cannot be written as $www$ for some $w \in \Sigma^*$, i.e., $a^{p+k}ba^pba^pb \notin A$, which contradicts the first property of the pumping lemma. Therefore, $A$ is a nonregular language.

Note that if you instead chose the string $s = a^pa^pa^p = a^{3p}$, you would not get a contradiction. This is because you could then choose $x$, $y$, $z$ with $y = a^3$, and for any $i \ge 0$, the pumped string is

$$xy^iz = a^{3p+3(i-1)} = a^{3(p+i-1)} = a^{p+i-1}a^{p+i-1}a^{p+i-i} \in A,$$

so the first property of the pumping lemma holds, and there is no contradiction.