Lecture Notes 11
Monte Carlo Simulation

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Outline

- Introduction
- Monte Carlo Integration
- Project Management
- Asset-Price Models
- Pricing Options
- Interest-Rate Models

Be sure to check course webpage for reading assignment and HW

Introduction

- Simulation often used to analyze systems that are not discrete-event systems
- Monte Carlo simulation
  - Basically used to evaluate complicated finite-dimensional integrals
  - Input/output relationships are known
- Examples:
  - Project management
  - Finance, e.g., pricing options
  - Cash flows
  - Marketing

Monte Carlo Simulation

- Suppose we want to compute

$$\mu = \int_a^b g(x)\,dx$$

  where $g(x)$ is complicated function.
- Suppose we can write

$$g(x) = h(x)f(x)$$

  where f is a density function with support $[a, b]$.
- Then

$$\mu = \int_a^b g(x)\,dx \;=\; \int_a^b h(x)\,f(x)\,dx \;=\; \mathcal{E}_f[h(X)]$$

  - $\mathcal{E}_f$ denotes expectation under density f
  - random variable X has density f

Basics of Monte Carlo Simulation

- **Goal:** Use simulation to estimate

$$\mu \;=\; \int_a^b h(x)\,f(x)\,dx \;=\; \mathcal{E}_f[h(X)]$$

- **General Approach:**
  - Generate $X_1, X_2, \ldots, X_n$ IID having density f.
  - Then $h(X_1), h(X_2), \ldots, h(X_n)$ IID
  - Apply classical statistics to $h(X_1), h(X_2), \ldots, h(X_n)$
  - Monte Carlo estimator of μ is

$$\widehat{\mu}(n) \;=\; \frac{1}{n}\sum_{k=1}^{n} h(X_k).$$

  $\widehat{\mu}(n) \approx \mu$ for large n, by law of large numbers.

Basic Algorithm for Monte Carlo Simulation

Algorithm: to use simulation to estimate

$$\mu \;=\; \int_a^b h(x)\, f(x)\, dx \;=\; \mathcal{E}_f[h(X)]$$

1. Generate $X_1, X_2, \ldots, X_n$ IID having density f.
2. Monte Carlo estimator of μ is
$$\widehat{\mu}(n) \;=\; \frac{1}{n} \sum_{k=1}^{n} h(X_k)$$
3. Approximate $100(1-\alpha)\%$ confidence interval for μ:
$$\left[\, \widehat{\mu}(n) \;\pm\; \frac{z_{1-\alpha/2}\, S(n)}{\sqrt{n}} \,\right]$$
where
$$S^2(n) \;=\; \frac{1}{n-1} \sum_{k=1}^{n} [h(X_k) - \widehat{\mu}(n)]^2$$

Monte Carlo Integration: Example

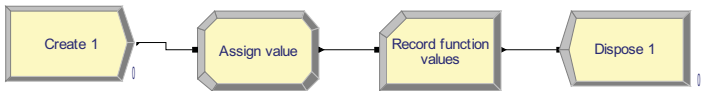
- Compute
$$\mu \;=\; \int_2^7 e^{\sin(x - \ln x)}\, dx$$
- Write μ as
$$\mu \;=\; \int_2^7 5 e^{\sin(x - \ln x)}\, \frac{1}{5}\, dx$$
so
$$\begin{aligned} h(x) &= 5 e^{\sin(x - \ln x)}, \\ f(x) &= (1/5) I_{[2,7]}(x) \;\sim\; \text{unif}[2, 7] \end{aligned}$$
- Thus, generate IID observations of
$$X \;\sim\; \text{unif}[2, 7],$$
and for each observation X, compute
$$h(X) \;=\; 5 e^{\sin(X - \ln X)}$$

Monte Carlo Integration: Excel/PopTools Implementation

	A	B	C	D	E	F	G	H	I	J	K
1	X	2.276373		PopTools	Mean	Variance	Lower CL	Upper CL	Valid iterations	Time taken	95% CI HW
2	h(X)	13.4988			7.0227	20.8763	1.84196	13.5831	10000	12 sec	0.0895536
3											
4				Yasai							
5				13.4988							

- Formulas
  - B1: `=2+5*rand()`
  - B2: `=5*EXP(SIN(B1-LN(B1)))`
  - J2: `=1.96*SQRT(E2/H2)`
- 95% C.I. from 10^4 replications with PopTools: 7.023 ± 0.090

Monte Carlo Integration: Arena Implementation



- Assign Module
  - Attribute Name: sample
    New Value: unif(2,7)
  - Attribute Name: function value
    New Value: 5*EP(SIN(sample-LN(sample)))
  - Use Expression Builder (right click in New Value slot)
- Record Module
  - Type: Expression
  - Value: function value
- 95% C.I. from 10^4 replications: 6.980 ± 0.091

Monte Carlo Integration: Remarks

- Suppose goal is to compute

$$\mu \;=\; \int_0^1 g(x)\,dx \;=\; \int_0^1 h(x)\,f(x)\,dx \;=\; \mathcal{E}_f[h(X)]$$

- **Trapezoid method** of numerical analysis: estimator is

$$\tilde{\mu}(n) \;=\; \sum_{i=1}^{n} \frac{g((i-1)/n) + g(i/n)}{2n}$$

  where we evaluated g at $n+1$ points $\frac{0}{n},\ \frac{1}{n},\ \frac{2}{n},\ \ldots,\ \frac{n}{n}$
  - When computing d-dimensional integral, error is $O(n^{-2/d})$ for trapezoid method.
- With sample size n, MC integration estimator has error $O_p(n^{-1/2})$, **independent of dimension** d.
  - MC integration is attractive when $d > 4$

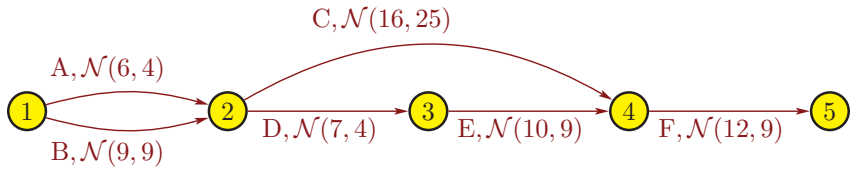
Project Management

- Company is going to build new widget
- Widget is produced by assembling part 1 and part 2
- Before production begins on either part 1 or 2,
  - Raw materials must be purchased
  - Workers must be trained
- Before parts 1 and 2 can be assembled into widget
  - Finished part 2 must be inspected
- Want to know if project takes less than 43 days to complete.
- **PERT/CPM**: Project Evaluation and Review Technique / Critical Path Method

PERT Network

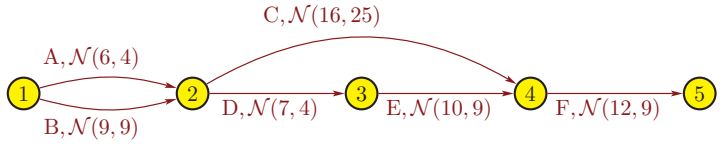
Activity	Predecessors	Duration (days)
A = train workers	—	$\mathcal{N}(6,4)$
B = purchase raw materials	—	$\mathcal{N}(9,9)$
C = produce part 1	A, B	$\mathcal{N}(16,25)$
D = produce part 2	A, B	$\mathcal{N}(7,4)$
E = test part 2	D	$\mathcal{N}(10,9)$
F = assemble parts 1 and 2	C, E	$\mathcal{N}(12,9)$

All activity durations are independent.



Note: $\mathcal{N}(a,b)$ = normal with mean a and variance b

PERT Network



- Time to complete project:

$$T \;=\; \text{Length of longest path in network}$$

  - Path 1 = Task A + Task C + Task F
  - Path 2 = Task A + Task D + Task E + Task F
  - Path 3 = Task B + Task C + Task F
  - Path 4 = Task B + Task D + Task E + Task F
- Expected time to complete project:

$$\mathcal{E}[T] \;=\; \mathcal{E}[\,\max(\text{Path 1, Path 2, Path 3, Path 4})\,]$$

PERT Network

- Expected time to complete project:

$$\begin{aligned}\mathcal{E}[T] &= \mathcal{E}[\,\max(\text{Path 1}, \ldots, \text{Path 4})\,] \\ &= \mathcal{E}[\,\max(A+C+F, \ldots, B+D+E+F)\,] \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \max(a+c+f, \ldots, b+d+e+f) \\ &\quad \times \phi_{(\mu_A,\sigma_A^2)}(a)\cdots\phi_{(\mu_F,\sigma_F^2)}(f)\,da\cdots df\end{aligned}$$

  where $\phi_{(\mu,\sigma^2)}(x)$ is $\mathcal{N}(\mu,\sigma^2)$ density fcn

  - Hence, $\mathcal{E}[T] = \mathcal{E}[h(A,B,\ldots,F)]$, where $h(a,b,\ldots,f) = \max(a+c+f,\ldots,b+d+e+f)$.

- Probability that project takes < 43 days to complete

$$\begin{aligned}\mathcal{P}(T<43) &= \mathcal{E}[\,I\{\,\max(\text{Path 1}, \ldots, \text{Path 4}) < 43\,\}\,] \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} I\{\,\max(a+c+f,\ldots,b+d+e+f) < 43\,\} \\ &\quad \times \phi_{(\mu_A,\sigma_A^2)}(a)\cdots\phi_{(\mu_F,\sigma_F^2)}(f)\,da\cdots df\end{aligned}$$

PERT Network

Monte Carlo simulation approach

- Generate observations of $A \sim \mathcal{N}(6,4), \ldots, F \sim \mathcal{N}(12,9)$
- Compute Path 1 $= A+C+F$, …, Path 4 $= B+D+E+F$
- Compute $T = \max(\text{Path 1}, \ldots, \text{Path 4})$ and $I(T<43)$
- Repeat n times, and compute average and variance of n outputs of T and $I(T<43)$ to get confidence intervals

Remarks

- In Arena, NORM(a,b) is normal with mean a and **std deviation** b.
- Beta distributions are often used instead of normals.
- $\exists$ efficient algorithm to find the longest path in PERT networks.

PERT Network: Excel/PopTools Implementation

	A	B	C	D	E	F	G	H	I	J
1	Activity	Mean	Variance	Time		Path	Time			
2	A	6	4	6.697075		1	36.06214			
3	B	9	9	5.592939		2	35.10517			
4	C	16	25	21.23087		3	34.95801			
5	D	7	4	5.562257		4	34.00104			
6	E	10	9	14.71164						
7	F	12	9	8.134199						
8										
9	Outputs			Mean	Variance	Lower CL	Upper CL	Valid iterations	Time taken	95% CI HW
10	Project Time	36.06214		40.37003	27.76201	30.42332	51.07561	10000	7 sec	0.10327176
11	P(T<43)	1		0.6965	0.211409	0	1	10000		0.00901193

- Formulas
  - D2: `=NORMINV(RAND(),B2,SQRT(C2))`
  - G2: `=D2+D4+D7`
  - B10: `=MAX(G2:G5)`
  - B11: `=IF(B10<43,1,0)`
- 95% C.I. from 10^4 replications with PopTools:
  - $E[T]$: 40.370 $\pm$ 0.103
  - $\mathcal{P}(T<43)$: 0.696 $\pm$ 0.009

PERT Network: Arena Implementation



- Assign Module
  - Attribute, Task A, NORM(6,2)
  - …
  - Attribute, Task F, NORM(12,3)
  - Attribute, Path 1, Task A + Task C + Task F
  - Attribute, Path 2, Task A + Task D + Task E + Task F
  - Attribute, Path 3, Task B + Task C + Task F
  - Attribute, Path 4, Task B + Task D + Task E + Task F
  - Attribute, Longest Path, MX(Path 1, Path 2, Path 3, Path 4)

PERT Network: Arena Implementation

- Record Module (if less than 43 Days)
  - Expression, Longest Path < 43
- Run 10^4 "replications"
- 95% C.I.'s
  - Expected Project Length:

$$40.504 \pm 0.108$$

  - P(Length $<$ 43 Days):

$$0.692 \pm 0.010$$

Models for Prices of Risky Asset



Key Properties:

- Price changes are proportional to current price (exponential growth)
- Prices are positive (limited liability)
- Efficient Market hypothesis

Random-Walk (RW) Model of Asset Prices

New price = Current price
× [1 + expected rate of return + random element]

- $S(t)$ = Asset price at time t (time unit is years)
- $S(t+\delta t)$ = Asset price at time $t+\delta t$ (δt is a small time interval)

$$S(t+\delta t) = S(t)\left[1 + \mu\,\delta t + \sigma\sqrt{\delta t}\,\epsilon_t\right]$$

$$= S(t) + \mu\,S(t)\,\delta t + \sigma\,S(t)\sqrt{\delta t}\,\epsilon_t$$

- μ = annualized expected rate of return in a short period of time
- σ = annual volatility of asset price
- $\epsilon_t \sim \mathcal{N}(0,1)$

Asset-Price Model: Example

- Microsquash is a computer software firm
  - No dividends to stockholders
  - Stock currently trades at \$100 per share
  - Annualized expected return $\mu = 15\%$
  - Annual volatility $\sigma = 30\%$
- Model for daily price change of stock
  - About 250 trading days per year, so use $\delta t = 1/250$ in

$$S(t+\delta t) = S(t)\left[1 + \mu\,\delta t + \sigma\sqrt{\delta t}\,\epsilon_t\right]$$

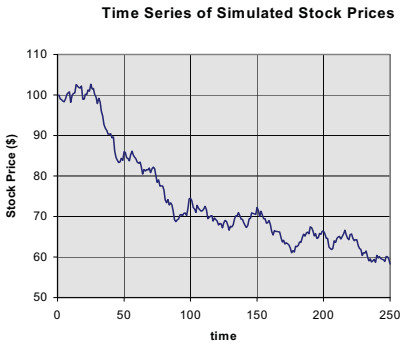
    to get next day's price of stock.
  - Given today's price $S(t)$, change in price tomorrow, $S(t)\left[\mu\,\delta t + \sigma\sqrt{\delta t}\,\epsilon_t\right]$, is normal
    - ▷ Mean $S(t)\,\mu\,\delta t$
    - ▷ Standard deviation $S(t)\,\sigma\sqrt{\delta t}$

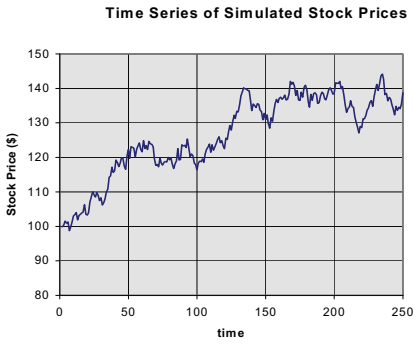
Asset-Price Model: Example

- Simulate stock price over 1 year with $\delta t = 1/250$
- Generate stock prices using

$$S(t+\delta t) \;=\; S(t)\left[1 \;+\; \mu\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_t\right]$$

- Examples of sample paths





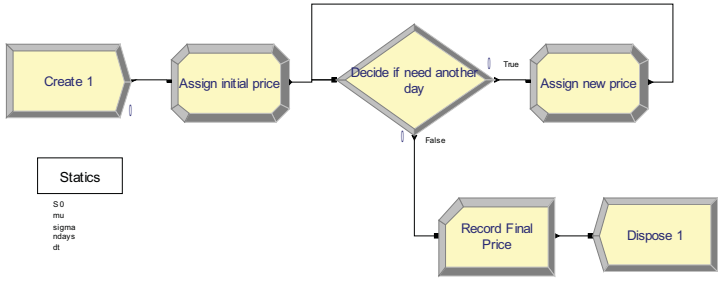
Asset-Price Model: Excel

	A	B	C
1	S(0)	100	
2	mu	0.15	
3	sigma	0.3	
4	t	1	
5	dt	0.004	
6			
7	Day	Price	
8	0	100	
9	1	103.9124	
10	2	102.9816	
11	3	97.74205	
12	4	97.88074	
13	5	99.0576	
255	247	157.5364	
256	248	160.0252	
257	249	157.8263	
258	250	157.261	
259			

Formulas

- B9: `=B8*(1 + B$2*B$5 + B$3*SQRT(B$5)*NORMSINV(RAND()))`

Asset-Price Model: Arena



- *Assign new price* module
  - Attribute: price
    price*(1 + mu*dt + sigma*sqrt(dt)*NORM(0,1))
  - Attribute: day
    day+1
- Statics module (Elements template): define constants

Random-Walk Model as $\delta t \to 0$

- Random-walk (RW) model

$$S(t+\delta t) \;=\; S(t)\left[1 \;+\; \mu\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_t\right]$$

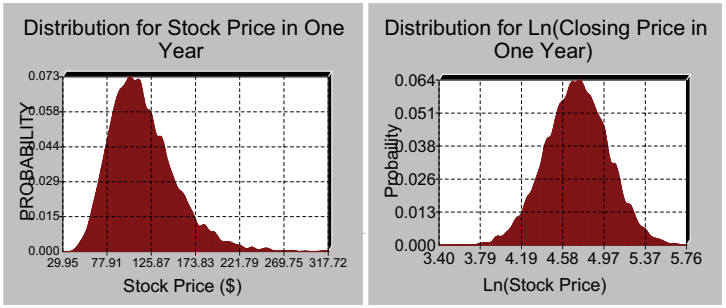
- Consider price at fixed time t, with $t = n\,\delta t$:

$$\begin{aligned} S(t) \;&=\; S(t-\delta t)\left[1 \;+\; \mu\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_{n-1}\right] \\ &=\; S(t-2\,\delta t)\left[1 \;+\; \mu\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_{n-2}\right] \\ &\quad\times\left[1 \;+\; \mu\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_{n-1}\right] \\ &=\; \cdots \\ &=\; S(0)\prod_{i=0}^{n-1}\left[1 \;+\; \mu\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_i\right] \end{aligned}$$

where $\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}$ are IID $\mathcal{N}(0,1)$

- What is distribution of $S(t)$ as $\delta t \to 0$ (so $n \to \infty$) ?

Distn of Asset Prices Using Random Walk Model



- In Arena, use Statistics module (Advanced Process template) to output data to filename.dat
- .dat file is binary, can only be opened by Arena's Output Analyzer
- start Output Analyzer from Windows
  - open New .dgr file, and add filename.dat file

Geometric Brownian Motion Model of Asset Prices

- If $X \sim \mathcal{N}(a, b^2)$, then $e^X \sim$ Lognormal(a, b^2)
- Recall that product of many independent random variables is approximately lognormal by CLT
- **Geometric Brownian motion (GBM) model** (AKA lognormal model) of asset prices:

$$\begin{aligned} S(t) &= S(0) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\sqrt{t}\, Z\right] \\ &= S(0)\, e^{X(t)} \end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$ and $X(t) \sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right) t,\ \sigma^2 t\right]$

GBM and RW With Small δt Are Consistent

- GBM: $S(T) = S(0)e^{X(T)}$ with $X(T) \sim \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right] T,\, \sigma^2 T\right)$.
- Then $X(T) = \ln(S(T)/S(0))$ and

$$\begin{aligned} \mathcal{E}[\ln(S(T)/S(0))] &= \left(\mu - \frac{\sigma^2}{2}\right) T, \\ \mathcal{V}[\ln(S(T)/S(0))] &= \sigma^2 T \end{aligned}$$

- Use simulation to compare random-walk (RW) model to GBM
  - Take $T = 1$ year, $\mu = 0.15$ and $\sigma = 0.3$, so $\mu - \frac{\sigma^2}{2} = 0.105$
  - Run many IID reps of $S(T)$ using RW model with $\delta t = 1/250$:

$$S(t + \delta t) = S(t)\left[1 + \mu\,\delta t + \sigma\sqrt{\delta t}\,\epsilon_t\right]$$

  - Compute sample mean and var of $\ln(S(T)/S(0))$ across reps

	Mean	Std Dev
$\ln(S(T)/S(0))$	0.1046	0.2984

Why is model called geometric Brownian motion?

- Let $W = [W(t) : t \geq 0]$ be a continuous-time stochastic process.
- **Defn:** W has **stationary increments** if distribution of increment $W(t+h) - W(t)$ depends only on $h \geq 0$ and not on t.
- **Defn:** W has **independent increments** if $s_1 \leq s_2 \leq t_1 \leq t_2$ implies $W(s_2) - W(s_1)$ is independent of $W(t_2) - W(t_1)$.
- **Defn:** W is a **Gaussian process** if its **finite-dimensional distributions** are multivariate normal $\mathcal{N}_d$
  - for every collection of $d \geq 1$ times $t_1, t_2, \ldots, t_d$,

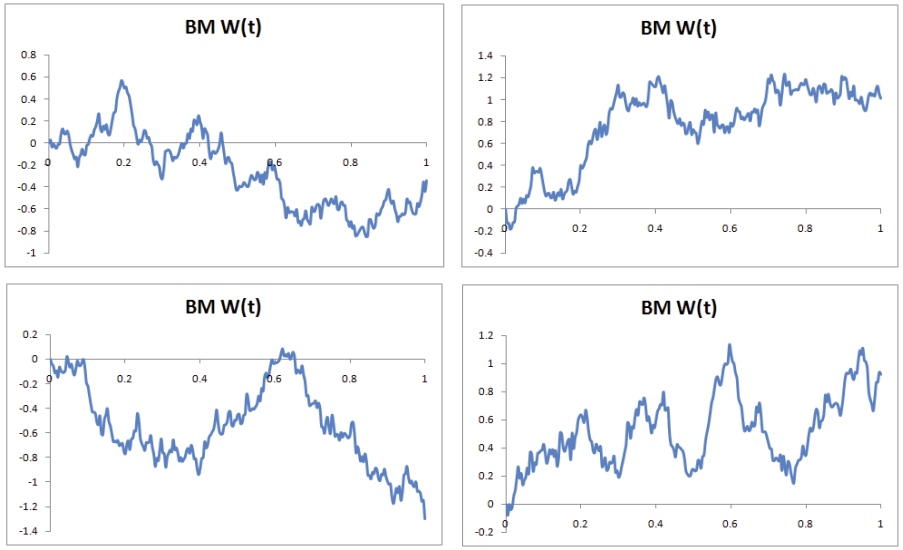
$$(W(t_1), W(t_2), \ldots, W(t_d)) \sim \mathcal{N}_d$$

- Gaussian process W defined by its
  - **mean function** $m(t) = \mathcal{E}[W(t)]$
  - **covariance function** $c(s,t) = \text{Cov}[W(s), W(t)]$.

Why is model called geometric Brownian motion?

- **Defn:** W is a (standard) **Brownian motion** (written $W \sim$ BM) if
  1. $W(0) = 0$
  2. W has stationary and independent increments
  3. For every $t \geq 0$ and $h > 0$,
$$W(t+h) - W(t) \;\sim\; \mathcal{N}(0, h)$$
- BM W is Gaussian process:
  - $W(t) = W(t) - W(0) \sim \mathcal{N}(0, t)$, so $m(t) = \mathcal{E}[W(t)] = 0$.
  - $c(s,t) = \text{Cov}[W(s), W(t)] = \min(s,t)$ since if $s \leq t$,
$$\begin{aligned} c(s,t) &= \mathcal{E}[\,W(s)\,W(t)\,] \;-\; \mathcal{E}[\,W(s)\,]\,\mathcal{E}[\,W(t)\,] \\ &= \mathcal{E}\big[\,W(s)\,[\,W(t) - W(s) + W(s)\,]\,\big] \;-\; 0 \\ &= \mathcal{E}\big[\,W(s)\,[\,W(t) - W(s)\,]\,\big] \;+\; \mathcal{E}[\,W^2(s)\,] \\ &= \mathcal{E}[\,W(s)\,]\,\mathcal{E}[\,W(t) - W(s)\,] \;+\; s \;=\; 0 \;+\; s \end{aligned}$$

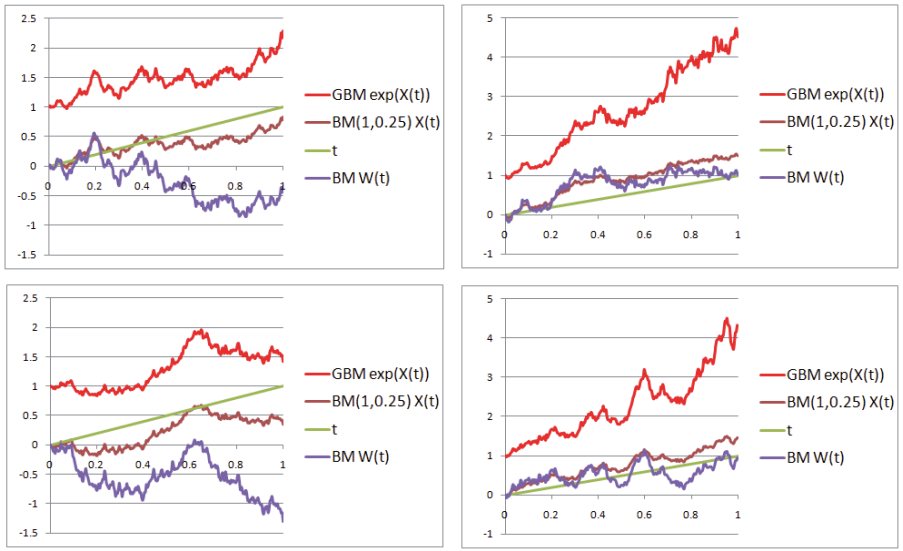
Plots of Standard Brownian Motion



Why is model called geometric Brownian motion?

- **Defn:** $X = [X(t) : t \geq 0]$ is **BM with drift a and diffusion coefficient** b^2 (written $X \sim \text{BM}(a, b^2)$) if
$$X(t) = at + bW(t) \quad \text{with} \quad W \sim \text{BM}.$$
  - $X(t) \sim \mathcal{N}(at,\, b^2 t)$ since $W(t) \sim \mathcal{N}(0,\, t)$.
  - Standard Brownian motion is $\text{BM}(0, 1)$.
- **Defn:** $Y = [Y(t) : t \geq 0]$ is **geometric Brownian motion (GBM)** if
$$Y(t) = e^{X(t)} \quad \text{with} \quad X \sim \text{BM}(a, b^2).$$
- Recall asset-price model
$$S(t) = S(0)e^{X(t)} \quad \text{with} \quad X(t) \sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right)t,\, \sigma^2 t\right]$$
  - $S(t) \sim \text{GBM}(\mu, \sigma^2)$ with drift and volatility parameters μ and σ.

Plots of BM, BM$(1, 0.25)$, and GBM



Stochastic Differential Equations

- By third part of definition of $W \sim$ BM,

$$W(t+\delta t) - W(t) \;\sim\; \mathcal{N}(0, \delta t)$$

- Recall our RW model:

$$\begin{aligned} S(t+\delta t) \;-\; S(t) \;&=\; S(t)\,\mu\,\delta t \;+\; S(t)\,\sigma\,\sqrt{\delta t}\,\epsilon_t \\ &\stackrel{\mathcal{D}}{=}\; S(t)\,\mu\,\delta t \;+\; S(t)\,\sigma\,[W(t+\delta t) - W(t)] \end{aligned}$$

since

- $\epsilon_t \sim \mathcal{N}(0,1)$ so $\sqrt{\delta t}\,\epsilon_t \sim \mathcal{N}(0, \delta t)$
- W has independent normal increments.

- Equating first and last lines and rearranging gives

$$\frac{S(t+\delta t) - S(t)}{S(t)} \;=\; \mu\,\delta t \;+\; \sigma\,[W(t+\delta t) \;-\; W(t)\,]$$

Stochastic Differential Equations

- Recall RW model satisfies

$$\frac{S(t+\delta t) - S(t)}{S(t)} \;=\; \mu\,\delta t \;+\; \sigma\,[W(t+\delta t) \;-\; W(t)\,]$$

- Letting $\delta t \to 0$ yields **stochastic differential equation (SDE)**

$$\frac{dS(t)}{S(t)} \;=\; \mu\,dt \;+\; \sigma\,dW(t)$$

- Can solve SDE (using Itô's formula) to get:

$$\begin{aligned} S(t) \;&=\; S(0)\,\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right] \\ &=\; S(0)\,e^{X(t)} \quad \text{with} \quad X \sim \text{BM}(\mu - \tfrac{\sigma^2}{2},\, \sigma^2), \end{aligned}$$

so $X(t) \sim \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right]t,\, \sigma^2 t\right)$

- Exactly GBM model of asset prices: *RW and GBM are consistent.*

GBM Model of Asset Prices: Excel Implmentation

	A	B	C
1	S(0)	100	
2	mu	0.15	
3	sigma	0.3	
4	t	1	
5			
6	S(t)	71.02567	
7			

- Recall GBM(μ, σ^2):

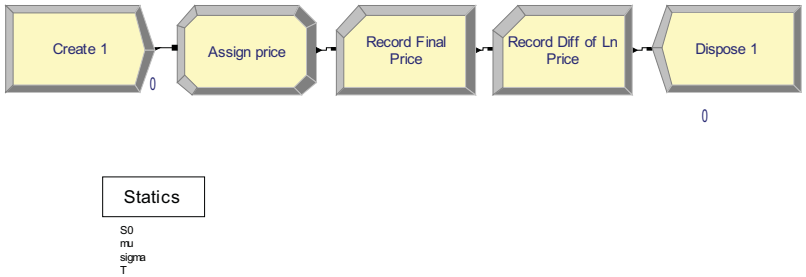
$$S(t) \;=\; S(0)\,e^{X(t)} \quad \text{with} \quad X \sim \text{BM}(\mu - \tfrac{\sigma^2}{2},\, \sigma^2), \text{ so}$$

$$X(t) \;\sim\; \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right]t,\, \sigma^2 t\right) \;\stackrel{\mathcal{D}}{=}\; \left[\mu - \frac{\sigma^2}{2}\right]t + \sigma\sqrt{t}\,\mathcal{N}(0,1)$$

- Excel formula for cell B6:

```
=B1*EXP((B2-B3^2/2)*B4+B3*SQRT(B4)*NORMSINV(RAND()))
```


GBM Model of Asset Prices: Arena Implmentation



- Assign price module
  - Attribute: price
    S0*ep((mu-0.5*sigma*sigma)*t + sigma*sqrt(t)*NORM(0,1))

Estimating Parameters of GBM Model

- GBM model: $S(t) = S(0)e^{X(t)}$ with $X \sim \text{BM}(\mu - \frac{\sigma^2}{2}, \sigma^2)$
- **Defn:** *Continuously compounded rate of return* is

$$\ln\left(\frac{S(t)}{S(0)}\right) \;=\; X(t) \;\sim\; \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right]t,\, \sigma^2 t\right)$$

- If t is one day ($\approx 1/250$ yrs), we can estimate μ and σ as follows:
  - Collect n observations $X_1, X_2, \ldots, X_n$, where $X_i = \ln(S_i/S_{i-1})$ and S_i is asset price on day i
  - Compute sample mean $\bar{X}_n$ and variance V_n of $X_1, X_2, \ldots, X_n$.
  - Set $\bar{X}_n = (\mu - \frac{\sigma^2}{2})t$ and $V_n = \sigma^2 t$ and solve for μ and σ to get estimates

$$\hat{\mu} \;=\; \frac{\bar{X}_n}{t} + \frac{V_n}{2t} \qquad \text{and} \qquad \hat{\sigma} \;=\; \sqrt{\frac{V_n}{t}}$$

Estimating Parameters of GBM Model

	A	B	C	D	E	F	G	H
1		GOOG	Daily Return					
2	Date	Adj Close	(S(t)-S(t-1))/S(t-1)	Ln(S(t)/S(t-1))			RW	GBM
3	31-Dec-08	307.65	0.014978061	0.014866997		n	253	253
4	30-Dec-08	303.11	0.019131195	0.018950495		dt	0.00395257	0.003953
5	29-Dec-08	297.42	-0.009788254	-0.009836474		average	-0.0026041	-0.0032
6	26-Dec-08	300.36	-0.008549266	-0.00858602		variance	0.00120923	0.001182
7	24-Dec-08	302.95	0.016542514	0.016407177		est of mu	-0.6588286	-0.66039
8	23-Dec-08	298.02	0.003062839	0.003058158		est of sigma	0.5531148	0.54677
9	22-Dec-08	297.11	-0.042105942	-0.043018094				
10	19-Dec-08	310.17	-0.000354518	-0.000354581				
11	18-Dec-08	310.28	-0.015734044	-0.015859138				
250	9-Jan-08	653.2	0.034067882	0.033500424				
251	8-Jan-08	631.68	-0.027061995	-0.027434914				
252	7-Jan-08	649.25	-0.011796043	-0.011866168				
253	4-Jan-08	657	-0.04133775	-0.042216456				
254	3-Jan-08	685.33	0.000204323	0.000204302				
255	2-Jan-08	685.19	-0.009096431	-0.009138056				
256	31-Dec-07	691.48						

Is GBM a good model for asset prices?

GBM: $S(t) \;=\; S(0)\, e^{X(t)}$ with $X \sim \text{BM}(\mu - \frac{\sigma^2}{2}, \sigma^2)$

1. Scaled Price Movement
   - Because GBM model follows from SDE

$$\frac{dS(t)}{S(t)} \;=\; \mu\, dt \;+\; \sigma\, dW(t),$$

   asset has identical **return** characteristics regardless of its price.
2. Positive Prices (limited liability)
   - Because asset prices are lognormal, they are always positive!
3. Efficient-Market Hypothesis
   - GBM is **Markov process**: current price of asset summarizes all information that can be used to predict its future prices.
   - Historical patterns do not provide any extra information.

Simulating GBM at Multiple Times

- Suppose we want to simulate asset prices

$$S(t) \;=\; S(0)\, e^{X(t)}$$

at m time points, $0 \equiv t_0 < t_1 < t_2 < \cdots < t_m$:

  - $X \sim \text{BM}\left(\mu - \frac{\sigma^2}{2},\; \sigma^2\right)$, so $X(t) \sim \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right]t,\; \sigma^2 t\right)$

- **Question:** How do we simulate this?
- **Answer:** Use *independent increments* property of BM (and GBM) to recursively simulate increments $X(t_i) - X(t_{i-1})$, $i = 1, 2, \ldots, m$.
  - For $i \neq j$, increments $X(t_i) - X(t_{i-1})$ and $X(t_j) - X(t_{j-1})$ are independent
  - $X(t_i) - X(t_{i-1}) \sim \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right](t_i - t_{i-1}),\; \sigma^2(t_i - t_{i-1})\right)$

Simulating GBM at Multiple Times

Algorithm: to simulate GBM at time points $0 \equiv t_0 < t_1 < \cdots < t_m$

1. For each $i = 1, 2, \ldots, m$, do the following:
2. Generate
$$X_i \sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}),\ \sigma^2(t_i - t_{i-1})\right],$$
where X_i is independent of $X_1, \ldots, X_{i-1}$.
3. Set
$$S(t_i) = S(t_{i-1})\, e^{X_i}$$

Important: It is **incorrect** to let

$$S(t_i) = S(0)\, e^{Z_i},\ i \geq 1, \text{ with } Z_i \sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right) t_i,\ \sigma^2 t_i\right] \text{ indep}$$

- Then $(S(t_0), S(t_1), \ldots, S(t_m))$ have wrong covariance.

Recap of Asset-Price Models: RW and GBM

- **Random-walk (RW) model**

$$S(t+\delta t) = S(t)\left[1 + \mu\,\delta t + \sigma\sqrt{\delta t}\,\epsilon_t\right] \quad \text{with} \quad \epsilon_t \sim \mathcal{N}(0,1)$$

  - **Approximation** for small δt
  - As $\delta t \to 0$, get **stochastic differential equation (SDE)**
$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW(t)$$

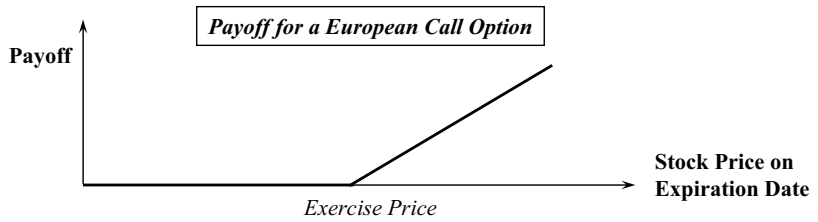
- Solving SDE yields **geometric Brownian motion** GBM(μ, σ^2)
$$S(t) = S(0)\, e^{X(t)} \quad \text{with} \quad X \sim \text{BM}(\mu - \tfrac{\sigma^2}{2},\ \sigma^2)$$
  - $X(t) \sim \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right] t,\ \sigma^2 t\right)$
  - **Exact** for all t, so should use GBM instead of RW model.

Options: Basic Terminology

- In finance, simulation often used to price **options**.
- **Call Option:** Gives holder the **right** (but not the obligation)
  - to **buy** underlying asset
  - by a specified date T (**expiration date** or **maturity**)
  - for a specified price K (**exercise price** or **strike price**).
- **Put Option:** Gives holder the right (but not the obligation)
  - to **sell** underlying asset
  - by a specified date T (expiration date or maturity)
  - for a specified price K (exercise price or strike price).
- **European Options:** Can only exercise **at** expiration T.
- **American Options:** Can exercise at any time **up to** T.

Options: Basic Terminology

- **Payoff** for European call option is
$$(S(T) - K)^+ \equiv \max(S(T) - K, 0)$$
  - $S(T)$ = price of underlying asset on expiration date T
  - K = strike (or exercise) price



- Example: European call option on IBM with strike $K = \$100$
  - If $S(T) = \$120$, exercise option and sell stock for \$20 profit.
  - If $S(T) = \$90$, don't exercise option, so \$0 profit.

Option Pricing: Risk-Neutral Valuation

- **Black-Scholes formula** for option pricing:

  Fair price of option
  $=$ Expected discounted option payoff
  in risk-neutral world.

- In a **risk-neutral world**, the expected return of underlying asset is the risk-free interest rate r.
  - Risk-free interest rate r is often taken to be the rate of a U.S. Treasury bill with the same maturity as the option.
  - If asset price $S(T)$ follows GBM, then in risk-neutral world,

    $$S(T) = S(0)\, e^{X(T)} \quad \text{with} \quad X(T) \sim \mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right]$$

  - **Key Idea:** When pricing options, asset's mean rate of return μ is replaced with risk-free interest rate r in distn for $X(T)$.

Option Pricing: Risk-Neutral Valuation

- **Recall:** to price option, simulate underlying asset value in risk-neutral world.
  - If asset price $S(T)$ follows GBM, then

    $$S(T) = S(0)\, e^{X(T)} \quad \text{with} \quad X(T) \sim \mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right]$$

  - Replaced asset's mean rate of return μ with risk-free interest rate r in $X(T)$.
- **Black-Scholes:** fair price p of European call option with strike price K and expiration T is

  $$p = \mathcal{E}[e^{-rT}(S(T) - K)^+],$$

  which is expectation of **discounted payoff** $e^{-rT}(S(T) - K)^+$, where $S(T)$ (and $X(T)$) defined as above.
- Why do we replace μ with r when pricing options?
  - μ is already reflected in the current asset price.
  - If we don't replace μ with r, then resulting price of option leads to *arbitrage opportunity* (risk-free profit).

Option Pricing: Example

- European call option with maturity at time T and strike price K
- To determine fair price p of option, we need
  - r = risk-free interest rate
  - σ = asset's annual volatility
  - $S(0)$ = asset price at the time of option purchase
  - $S(T)$ = asset price (in risk-neutral world) on the maturity date under GBM:

    $$S(T) = S(0)\, e^{X(T)} \quad \text{with} \quad X(T) \sim \mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right]$$

Option Pricing: Example

- Recall: asset price (in risk-neutral world) under GBM:

  $$S(T) = S(0)\, e^{X(T)} \quad \text{with} \quad X(T) \sim \mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right]$$

- Black-Scholes model gives the fair price p of call option as expected discounted payoff in a risk-neutral world under GBM:

  $$\begin{aligned} p &= \mathcal{E}\left[e^{-rT}\,(S(T) - K)^+\right] \\ &= \mathcal{E}\left[e^{-rT}\,(S(0)e^{X(T)} - K)^+\right] \\ &= \int_{-\infty}^{\infty} e^{-rT}\,(S(0)\,e^x - K)^+\,\phi(x)\,dx \end{aligned}$$

  - ϕ is density function of $\mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right]$
  - Fair price $p = \mathcal{E}[h(X)]$, where $h(x) = e^{-rT}(S(0)e^x - K)^+$ is discounted payoff function and $X \sim \phi$

Simulation Procedure

1. Simulate asset prices using GBM in risk-neutral world.
   - Given asset price $S(0)$ at time 0, generate asset price at time T as

$$S(T) = S(0)\, e^{X(T)} \quad \text{with} \quad X(T) \sim \mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right]$$

2. Compute discounted option payoff using simulated prices; i.e., multiply the option payoff by e^{-rT}
   - For European call option, discounted payoff is

$$e^{-rT}(S(T) - K)^+$$

3. Repeat the simulation many times.
4. Estimated option price = avg of discounted payoffs from step 2.

Example: Pricing European Call Option

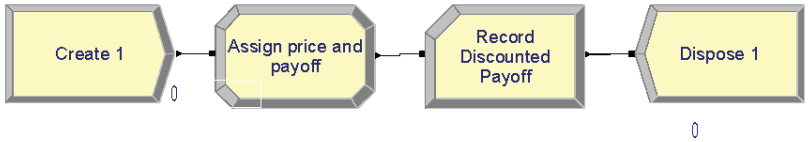
- **European** call option on MSFT:
  - Stock price was \$60.66 on August 22, 2001.
  - Expiration date: 28 trading days
  - Strike price: $K = \$55$
  - Risk-free rate of return: $r = 3.27\%$ (3-month Treasury rate)
  - Annualized std dev of returns: $\sigma = 52\%$ per year.
  - Assume MSFT stock price behaves as GBM.
- Goals:
  - Use Monte Carlo simulation to estimate fair price for this option.
  - Compare to the price of option obtained by Black-Scholes formula.
  - **American** option traded for \$6.20 on August 22, 2001. Compare this to price obtained from Black-Scholes formula.

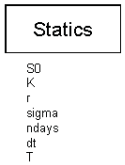
Pricing European Call Option: Excel/PopTools

	A	B	C	D	E	F	G	H	I	J
1	S(0)	$ 60.66								
2	r	0.0327								
3	sigma	0.52								
4	K	55								
5	dt	0.004								
6	days	28								
7	T	0.112								
8				Mean	Variance	Lower CL	Upper CL	Valid iterations	Time taken	95% CI HW
9	S(T)	$ 78.21		60.78219	112.6072	42.5648	84.1652	10000	5 sec	0.207988372
10	discounted payoff	$ 23.12		7.522343	72.5211	0	29.0586	10000		0.16691227

- Formulas
  - B7: `=B5*B6`
  - B9: `=B1*EXP((B2-B3^2/2)*B7+B3*SQRT(B7)*NORMSINV(RAND()))`
  - B10: `=EXP(-B2*B7)*MAX(B9-B4,0)`
- 95% CI from 10^4 replications using PopTools: 7.52 ± 0.17
  - Thus, 95% confident that fair price for option lies between \$7.35 and \$7.69

Pricing European Call Option: Arena Implementation

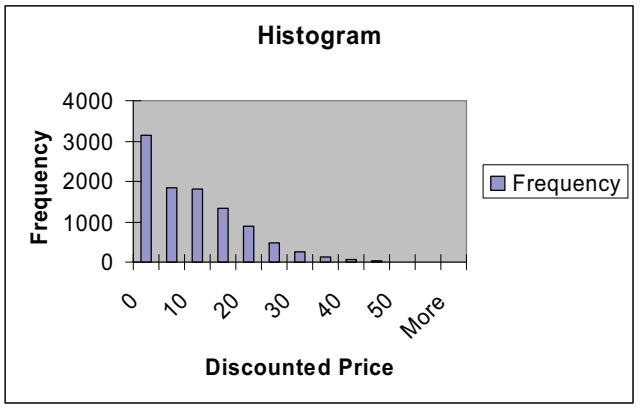




Arena Model for Pricing Option

- Assign Module
  - Attribute Name: price
    New Value:
    S0*ep((r-0.5*sigma*sigma)*T+sigma*sqrt(T)*NORM(0,1))
  - Attribute Name: payoff
    New Value: MAX(price-K,0)
- Record Module
  - Type: Expression
    Value: ep(-r*T)*payoff
- Statics Module (from Elements template): Can use this to define constants

Arena Model for Pricing Option



Average	HW	Min	Max	Obs
7.567	0.18	0	52.49	10000

Black-Scholes Formula

- No need to simulate to determine fair price for this option.
- Black-Scholes formula gives fair price of European call option as

$$p = \int_{-\infty}^{\infty} e^{-rT} \, (S(0)\, e^x - K)^+ \, \phi_{(r-\frac{\sigma^2}{2})T, \sigma^2 T}(x) \, dx$$

$$= S(0)\, \Phi(d_1) - K\, e^{-rT} \Phi(d_2)$$

where $\Phi \equiv \Phi_{0,1}$ is CDF of $\mathcal{N}(0,1)$,

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

  - In Excel, can use `normsdist` to numerically compute CDF Φ
- For our example, Black-Scholes formula results in price of $7.57

Pricing Exotic Options

- **Exotic options**: more complicated than *vanilla options* from before.
  - Payoff may depend on **entire path** of underlying asset's price and not just price at expiration (as in vanilla options).
  - Fair price p of exotic option is expected discounted payoff (under risk-neutral valuation).
  - Often, no closed-form formula (such as Black-Scholes) for p, so must use other methods, e.g., simulation.
- **Example:** European **knockout** (or **barrier**) call option
  - Payoff depends on whether asset price goes outside specified **barriers** (A, B) at specified discrete times $t_1, t_2, \ldots, t_m = T$
  - If asset price goes outside barrier at some time t_i, payoff is 0.
  - Otherwise, payoff at expiration T is $(S(T) - K)^+$
  - No closed-form formula for p for discretely-monitored barrier.

Example: European Knockout Call Option

- Expiration in T years, which is m days
  - $\Delta T =$ one day $\approx 1/250$;
    $T = m\,\Delta T$; $t_i = i\Delta T$ is end of ith day.
- $K =$ strike price
- $A =$ lower barrier, $B =$ upper barrier
- $S(t) =$ price of underlying asset at time t; given $S(0)$.
- $\sigma =$ annual volatility
- $r =$ risk-free rate
- Payoff: $I\{A < S(t_i) < B, \forall\, i \le m\}\,(S(T) - K)^+$
  - If $A < S(t_i) < B$ for all $i = 1, 2, \ldots, m$,
    then payoff is $(S(T) - K)^+$
  - Otherwise, payoff is 0.

Example: European Knockout Call Option

- Assume price of underlying asset follows GBM in risk-neutral world:

$$S(t_i) = S(t_{i-1})\, e^{X_i}$$

  - $X_1, \ldots, X_m \sim \mathcal{N}\left[\left(r - \frac{\sigma^2}{2}\right)\Delta T,\ \sigma^2 \Delta T\right]$ IID
  - Asset price on ith day

$$\begin{aligned} S(t_i) &= S(t_{i-1})\, e^{X_i} \\ &= S(t_{i-2})\, e^{X_{i-1}}\, e^{X_i} = \cdots \\ &= S(0) \prod_{j=1}^{i} e^{X_j}, \end{aligned}$$

which is a function of i IID (non-standard) normals $X_1, \ldots, X_i$

Example: European Knockout Call Option

- **Discounted payoff** of knockout option

$$\begin{aligned} & e^{-rT} I\{A < S(t_i) < B, \forall\, i \le m\}\,(S(T) - K)^+ \\ &= e^{-rT} I\left\{A < S(0) \prod_{j=1}^{i} e^{X_j} < B, \forall\, i \le m\right\}\left(S(0) \prod_{j=1}^{m} e^{X_j} - K\right)^+ \\ &\equiv h(X_1, \ldots, X_m) \end{aligned}$$

  - $X_1, \ldots, X_m$ IID with density ϕ_{γ,ψ^2} of $\mathcal{N}(\gamma, \psi^2)$
  - $\gamma = \left(r - \frac{\sigma^2}{2}\right)\Delta T$ is mean of X_i, and $\psi^2 = \sigma^2 \Delta T$ is variance
- **Fair price** p of knockout option is **expected discounted payoff**

$$\begin{aligned} p &= \mathcal{E}[h(X_1, \ldots, X_m)] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(x_1, \ldots, x_m) \prod_{j=1}^{m} \phi_{\gamma,\psi^2}(x_j)\, dx_j \end{aligned}$$

which is m-dimensional integral, where m is number of days.

Algorithm: European Knockout Call Option

1. On each independent replication $k = 1, 2, \ldots, n$,
   - Generate $X_{k,1},\ X_{k,2}, \ldots,\ X_{k,m} \sim \mathcal{N}(\gamma, \psi^2)$ IID.
   - Compute discounted payoff $D_k = h(X_{k,1}, \ldots, X_{k,m})$, where

$$\begin{aligned} h(X_{k,1}, \ldots, X_{k,m}) = e^{-rT} I\left\{A < S(0) \prod_{j=1}^{i} e^{X_{k,j}} < B, \forall\, i \le m\right\} \\ \times \left(S(0) \prod_{j=1}^{m} e^{X_{k,j}} - K\right)^+ \end{aligned}$$

2. Compute 95% CI for p as $[\bar{D}_n \pm 1.96 S_n/\sqrt{n}\,]$, where
   $\bar{D}_n = \frac{1}{n}\sum_{k=1}^{n} D_k$ and $S_n^2 = \frac{1}{n-1}\sum_{k=1}^{n}(D_k - \bar{D}_n)^2$.

Remark: More efficient to compute $S(t_i) = S(0)\exp(\sum_{j=1}^{i} X_j)$ and $I\left\{\ln[A/S(0)] < \sum_{j=1}^{i} X_{k,j} < \ln[B/S(0)]\right\}$;
precompute constants; only compute payoff if $I\{\cdots\} = 1$.

Knockout Option: Example

Find fair price p for European knockout call option

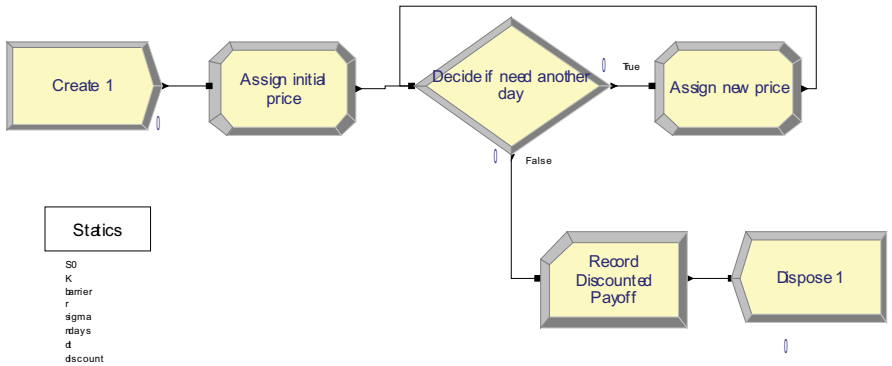
- $T = 0.12$ (i.e., 30 trading days)
  - $\Delta T = 1/250$ (one day)
- $K = \$40$ (strike price)
- $A = \$37$ (lower barrier)
- $B = +\infty$ (no upper barrier)
- $S(0) = \$42$ (current price of underlying stock)
- $\sigma = 30\%$ (stock's annual volatility)
- $r = 10\%$ (risk-free rate)

Knockout Option: Excel/PopTools Implementation

	A	B	C	D	E	F	G	H	I	J
1	S(0)	42.00								
2	r	0.1								
3	sigma	0.3								
4	K	40								
5	barrier	37								
6	dt	0.004								
7	days	30								
8	T	0.12								
9										
10	day	asset price								
11	0	42.00								
12	1	42.50								
13	2	42.59								
40	29	45.40								
41	30	45.21		Mean	Variance	Lower CL	Upper CL	Valid iterations	Time taken	95% CI HW
42	discounted payoff	5.144084498		3.228055	12.30811	0	11.64431	10000	26 sec	0.068762524

- Formulas
  - B12: `=B11*EXP((B$2-B$3^2/2)*B$6+B$3*SQRT(B$6)*NORMSINV(RAND()))`
  - B42: `=IF(MIN(B11:B41)<B5,0,EXP(-B2*B8)*MAX(B41-B4,0))`
- 95% CI from 10^4 replications: 3.23 ± 0.07

Knockout Option: Arena Implementation



Knockout Option: Arena Implementation

- Statics Module:

Static	Default Value
dt	1/250
S0	42
K	40
barrier	37
ndays	30
sigma	0.30
r	0.10
discount	ep(-r*ndays*dt)

- Assign Module:

Attribute	New Value
price	S0
inbounds	1

Knockout Option: Arena Implementation

- Assign Module
  - Attribute Name: price
    New Value:
    price*ep((r-0.5*sigma*sigma)*dt+sigma*sqrt(dt)*NORM(0,1))
  - Attribute Name: inbounds
    New Value: inbounds*(price>barrier)
- Record Module for discounted payoff
  - Type : Expression
    Expression: discount*inbounds*MAX(price-K,0)
- Results:

Name	Average	HW	Min	Max	Obs
discounted payoff	3.26	0.18	0	24.21	2000

  - Fair price of option is between \$3.08 and \$3.44

Summary of GBM and Options-Pricing Simulations

- Price of underlying asset is GBM(μ, σ^2) with drift parameter μ and volatility parameter σ:

$$S(t) \;=\; S(0)\, e^{X(t)} \quad \text{with} \quad X \;\sim\; \text{BM}\left(\mu - \tfrac{\sigma^2}{2},\, \sigma^2\right)$$

  so $X(t) \sim \mathcal{N}\left(\left[\mu - \tfrac{\sigma^2}{2}\right] t,\; \sigma^2 t\right)$

- Fair price of option
  = Expected discounted option payoff in risk-neutral world
  - Replace asset's mean return rate μ with risk-free interest rate r
- Can determine option prices using simulation

Asset-Price Model: Jump-Diffusion Model

- Merton introduced **jump-diffusion model** for asset prices that allows for occasional jumps in price
  - Jumps model "shocks" to asset
- Suppose that asset price jumps at random times $\tau_1, \tau_2, \ldots$
- At jump time τ_j
  - Size of jump is positive RV Y_j, so price is

$$S(\tau_j) \;=\; Y_j\, S(\tau_j-)$$

  - If $Y_j > 1$, then price jumps up.
  - If $Y_j < 1$, then price jumps down.
- Between jump times, asset price $S(t)$ evolves as ordinary GBM.
- Assume GBM, inter-jump times and jump sizes are independent.

Asset-Price Model: Jump-Diffusion Model

- What are asset's prices at random **jump times** $\tau_1, \tau_2, \ldots$?
  - Given price $S(\tau_{j-1})$ immediately after $(j-1)$th jump.
  - Until next jump (at time τ_j), price evolves as GBM

$$S(\tau_j-) \;=\; S(\tau_{j-1})\, e^{X_j}$$

    with $X_j \;\sim\; \mathcal{N}[(\mu - \frac{\sigma^2}{2})(\tau_j - \tau_{j-1}),\, \sigma^2(\tau_j - \tau_{j-1})]$
  - At time τ_j, jump of size Y_j occurs:

$$S(\tau_j) \;=\; Y_j\, S(\tau_j-)$$

- Asset's price at **fixed time** t is

$$S(t) \;=\; S(0)\, e^{X(t)} \prod_{j=1}^{A(t)} Y_j \quad \text{with} \quad X \;\sim\; \text{BM}(\mu - \tfrac{\sigma^2}{2},\; \sigma^2)$$

  - $X(t) \;\sim\; \mathcal{N}\left(\left[\mu - \tfrac{\sigma^2}{2}\right] t,\; \sigma^2 t\right)$
  - $A(t) \;=\;$ number of jumps by time t

Asset-Price Model: Jump-Diffusion Model

- Suppose we want to simulate asset prices

$$S(t) \;=\; S(0)\, e^{X(t)} \prod_{j=1}^{A(t)} Y_j$$

at m **fixed** time points, $0 \equiv t_0 < t_1 < t_2 < \cdots < t_m$:

  - $X \sim$ BM $\left(\mu - \frac{\sigma^2}{2},\, \sigma^2\right)$, so $X(t) \sim \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right] t,\, \sigma^2 t\right)$
  - $A(t)$ is **Poisson process** with rate λ (could instead use NHPP)
  - $Y_j \sim \mathcal{LN}(a, b^2)$ (could instead use any positive RV)
  - $X, A(t), Y_1, Y_2, \ldots$ are mutually independent.

- How do we simulate this?
- Since Poisson process $A(t)$ has **independent increments**,
  - $A(t_i) - A(t_{i-1})$ and $A(t_j) - A(t_{j-1})$ are independent, $i \neq j$
  - $A(t_i) - A(t_{i-1}) \sim$ Poisson$[\lambda(t_i - t_{i-1})]$

Asset-Price Model: Jump-Diffusion Model

Algorithm: to simulate jump-diffusion model at fixed times $0 \equiv t_0 < t_1 < \cdots < t_m$

1. For each $i = 1, 2, \ldots, m$, do the following
2. Generate $X_i \sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}),\; \sigma^2(t_i - t_{i-1})\right]$
3. Generate # jumps in $(t_{i-1}, t_i]$ as $A_i \sim$ Poisson$[\lambda(t_i - t_{i-1})]$
4. Generate jump sizes $Y_{i,1}, Y_{i,2}, \ldots, Y_{i,A_i} \sim \mathcal{LN}(a, b^2)$
5. Set

$$S(t_i) \;=\; S(t_{i-1})\, e^{X_i} \prod_{j=1}^{A_i} Y_{i,j}$$

where empty product $= 1$

Remark: All X_i, A_i, $Y_{i,j}$ are mutually independent.

Option Pricing for Jump-Diffusion Model

- Recall jump-diffusion model of asset prices:

$$S(t) \;=\; S(0)\, e^{X(t)} \prod_{j=1}^{A(t)} Y_j$$

  - $X \sim$ BM $\left(\mu - \frac{\sigma^2}{2},\; \sigma^2\right)$, so $X(t) \sim \mathcal{N}\left(\left[\mu - \frac{\sigma^2}{2}\right] t,\, \sigma^2 t\right)$
  - $A(t)$ is Poisson process with rate λ, and $Y_j \sim \mathcal{LN}(a, b^2)$
  - $X, A(t), Y_1, Y_2, \ldots$ are mutually independent.

- To price option on this asset, how do we do **risk-neutral valuation**?
  - Replace μ with $r - \lambda\nu$, so $X \sim$ BM $\left(r - \lambda\nu - \frac{\sigma^2}{2},\; \sigma^2\right)$, and $X(t) \sim \mathcal{N}\left(\left[r - \lambda\nu - \frac{\sigma^2}{2}\right] t,\; \sigma^2 t\right)$.
  - $r =$ risk-free interest rate.
  - $\nu = \mathcal{E}[Y_j - 1] = \exp\{a + b^2/2\} - 1 = \mathcal{E}[\text{jump size}] - 1$.

Interest-Rate Models

- Simulation models for stochastic interest rate
- $R(t) =$ interest rate at time t
- We consider three models:
  - Rendleman and Bartter Model
  - Vasicek Model
  - Cox, Ingersoll, and Ross Model
- Application: Simulating cash flows for a home mortgage loan.
- Objective:
  - Structure simulation models for interest-rate risk
  - Compare mortgage loans using simulation

Rendleman-Bartter Model

- Assumes interest rate $R(t)$ behaves like asset prices:

$$R(t+\delta t) \;=\; R(t) \;+\; \mu\, R(t)\, \delta t \;+\; \sigma\, R(t)\, \sqrt{\delta t}\, \epsilon_t$$

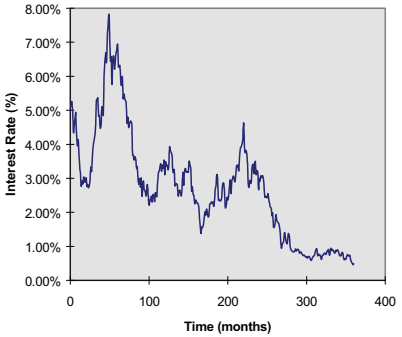
  - $\epsilon_t \sim \mathcal{N}(0,1)$
  - $\mu =$ (constant) mean instantaneous rate of change in interest rate
  - $\sigma =$ annual volatility of interest rate
  - $\delta t =$ small time step
- $R(t)$ satisfies SDE

$$\frac{dR(t)}{R(t)} \;=\; \mu\, dt \;+\; \sigma\, dW(t) \qquad \text{with} \qquad W \sim \text{BM},$$

  which we can solve to get $R(t)$ is GBM.
- Interest rates change more rapidly when they are high.
- Interest rate volatility is higher when interest rates are high.

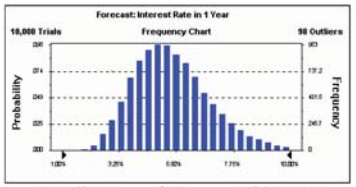
Rendleman-Bartter Model: Example

- Suppose $\mu = 0.05$, $\sigma = 0.30$, interest rate at year 0 $= 5.125\%$
- One sample path

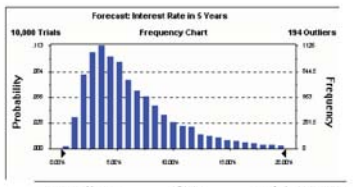


- Remark: Notice how volatility depends on interest rates. When rate is high, volatility also tends to be high.

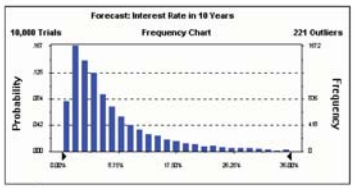
Rendleman-Bartter Model: Example



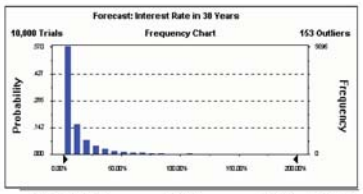
Median	*SD*	*5% VAR*
5.18%	1.56%	8.18%



Median	*SD*	*5% VAR*
5.30%	4.70%	15.69%



Median	*SD*	*5% VAR*
5.38%	10.10%	25.65%

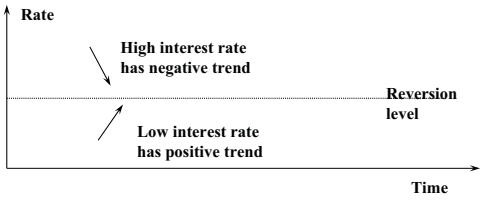


Median	*SD*	*5% VAR*
6.03%	64.41%	85.50%

5% *VAR* (Value-at-risk) is the number x s.t. $\mathcal{P}(R(t) > x) = 0.05$

Mean Reversion

- Over time, interest rates appear to be pulled back to some long-run average level.
- This phenomenon is known as **mean reversion**.
- Causes of mean reversion:
  - When rates are high, economy tends to slow down. Less demand for funds from borrowers, and rates drop.
  - When rates are low, more demand for funds and rates tend to increase.



Vasicek Model

- Vasicek model incorporates mean reversion:

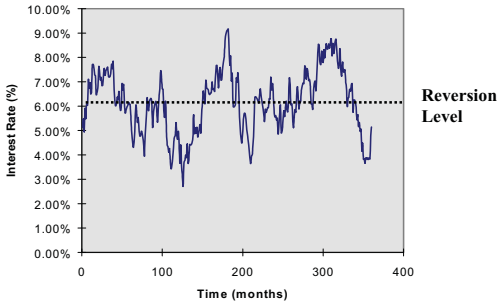
$$R(t+\delta t) \;=\; R(t) \;+\; a\,[b - R(t)]\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_t$$

- Vasicek model written as SDE:

$$dR(t) \;=\; a\,[b - R(t)]\,dt \;+\; \sigma\,dW(t) \quad \text{with} \quad W \sim \text{BM}.$$

- Remarks:
  - b = long-run equilibrium value of interest rate
  - parameter a governs speed of adjustment
    - ▷ Need $a > 0$ to insure stability around long-term value
  - Interest rates above reversion level have negative trend.
  - Interest rates below reversion level have positive trend.
  - Magnitude of trend depends on distance from reversion level.
  - Std. deviation of rate change is constant.

Vasicek Model



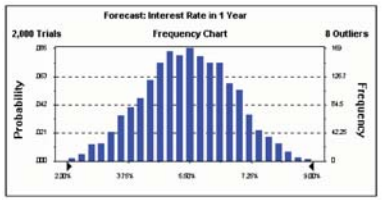
- Vasicek model:

$$R(t+\delta t) \;=\; R(t) \;+\; a\,[b - R(t)]\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_t,$$

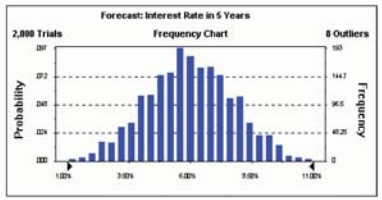
with $a = 0.36$, $b = 6.2\%$, $\sigma = 1.52\%$, $R(0) = 5.125\%$

- Do you see how this model captures mean reversion?
- Can you make any predictions about the mean and std. deviation of the interest rates 1, 5, 10 and 30 years from now?

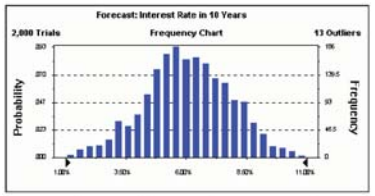
Vasicek Model



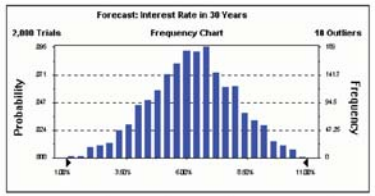
Mean	SD	5% V-A-R
5.43%	1.28%	7.52%



Mean	SD	5% V-A-R
5.98%	1.76%	8.97%



Mean	SD	5% V-A-R
6.08%	1.83%	9.01%



Mean	SD	5% V-A-R
6.23%	1.80%	9.15%

Cox-Ingersoll-Ross (CIR) Model

- In theory, Vasicek model

$$R(t+\delta t) \;=\; R(t) \;+\; a\,[b - R(t)]\,\delta t \;+\; \sigma\sqrt{\delta t}\,\epsilon_t$$

allows negative interest rates.

- Cox, Ingersoll and Ross (CIR) model avoids this problem:

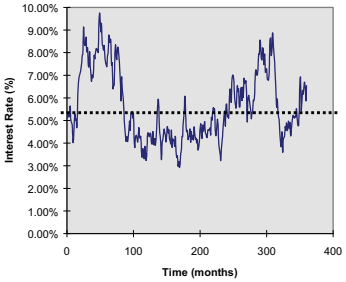
$$R(t+\delta t) \;=\; R(t) \;+\; a\,[b - R(t)]\,\delta t \;+\; \sigma\sqrt{R(t)}\,\sqrt{\delta t}\,\epsilon_t$$

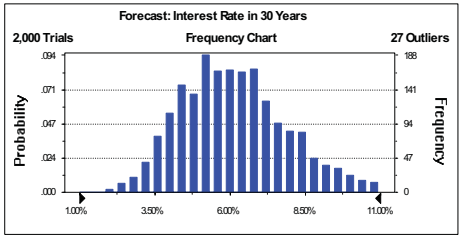
- CIR model written as SDE:

$$dR(t) \;=\; a\,[b - R(t)]\,dt \;+\; \sigma\sqrt{R(t)}\,dW(t) \quad \text{with} \quad W \sim \text{BM}.$$

- Main difference between Vasicek and CIR models
  - In CIR, magnitude of random part depends on current interest rate.
  - This ensures that interest rates cannot be negative (as $\delta t \to 0$).

Cox-Ingersoll-Ross Model





Mean	*SD*	*5% V-A-R*
6.19%	1.86%	9.52%

Application: Comparing Home Mortgages

- Suppose you are shopping for a mortgage loan for your new house. Need to borrow $100,000
- Considering four loans (all with zero closing points):
  - 30-year fixed:
    - ▷ Interest rate will be 8.00% for lifetime (30 years) of mortgage.
  - Adjustable:
    - ▷ Interest rate for first year will be 5.50%
    - ▷ Interest rate will be adjusted once every year and will be equal to the interest rate for 1-year Treasury bill plus a margin of 2.00%.

Application: Comparing Home Mortgages

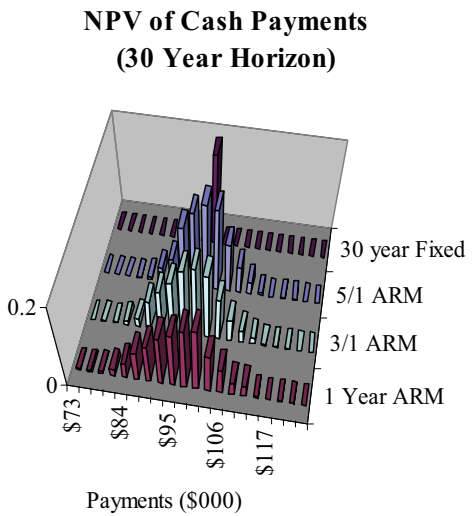
- 3/1 Fixed - Adjustable:
  - ▷ Interest rate will be 6.875% for first 3 years
  - ▷ Then convert to adjustable loan
- 5/1 Fixed - Adjustable:
  - ▷ Interest rate will be 7.50% for first 5 years.
  - ▷ Then convert to adjustable loan

Application: Comparing Home Mortgages

Interest-Rate Model:

- 1-year Treasury bill rate follows Vasicek process
  - $a = 0.36$
  - $b = 6.20\%$ (Reversion level)
  - $\sigma = 1.52\%$
- Current rate $= 5.125\%$
- Simulate *net present value* (NPV) of payments (using constant 8% discount rate) of four mortgages.

Home Mortgages: Simulation Results for Vasicek Model

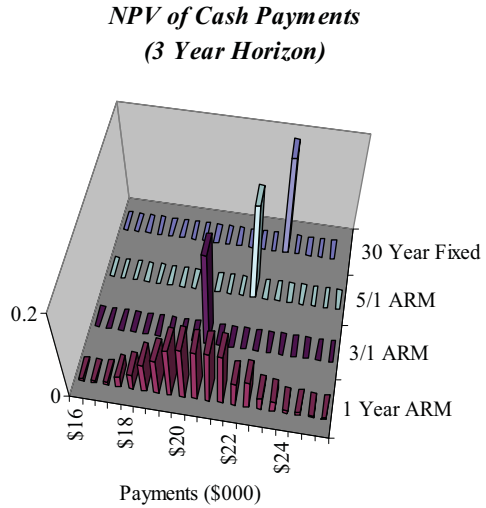


Home Mortgages: Simulation Results for Vasicek Model

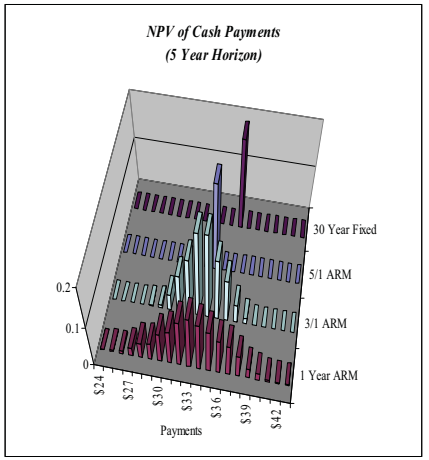
	NPV of Loan Payments		
	Mean	Std Dev	2.5% VaR
30-yr fixed	$100,000	—	$100,000
ARM	$97,705	$7,240	$112,221
3/1 ARM	$97,859	$6,299	$110,383
5/1 ARM	$98,929	$5,191	$109,242

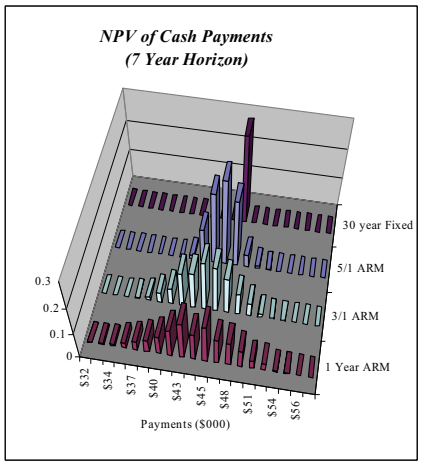
What do you conclude?

Home Mortgages: Simulation Results for Vasicek Model



Home Mortgages: Simulation Results for Vasicek Model





Home Mortgages: Simulation Results for R-B Model

	NPV of Loan Payments		
	Mean	Std Dev	2.5% VaR
30-yr fixed	\$100,000	—	\$100,000
ARM	\$96,075	\$47,022	\$206,726
3/1 ARM	\$96,774	\$45,765	\$202,443
5/1 ARM	\$98,540	\$43,397	\$197,646

What explains huge difference between VaR results from Vasicek model and Rendleman-Bartter model?

Summary of Interest-Rate Models

- Three models for interest rate risk
  - Rendleman-Bartter model
    - ▷ Interest rates behave like asset prices
  - Vasicek model
    - ▷ Interest rates satisfy mean reversion
  - Cox-Ingersoll-Ross model
    - ▷ Mean reversion and proportional rate changes
- Can be used to simulate cash flows that depend on interest rates.
- Results/conclusions are sensitive to assumptions of underlying model.