Lecture Notes 10
Simulation Output Analysis

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Outline

- Introduction
- Performance Measures
  - Transient (AKA Terminating)
  - Steady-State (AKA Infinite-Horizon)
- Output Analysis for Transient Simulations
- Output Analysis for Steady-State Simulations
- Other Useful Methods

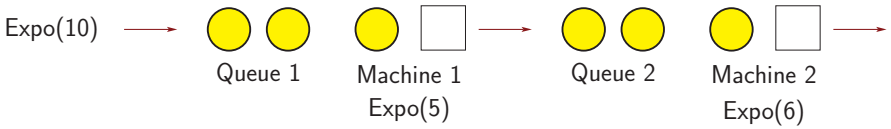
Be sure to check course webpage for reading assignment and HW

Introduction

- So you've finally finished developing your simulation model!
- Spent countless hours
  - understanding underlying processes
  - collecting data
  - fitting probability distributions
  - selecting a performance measure
  - coding/debugging simulation program
- Ran simulation program once
  - Seemed to indicate system would perform well if implemented.
- Once in place, system performed poorly
  - Not at all like results from the one simulation run.
- So what went wrong?

Example

- Manufacturing system



- μ = long-run average flow time
- Run one simulation of first 1000 jobs

$$\widehat{\mu} = 19.1$$

Introduction

- Many simulations include randomness
  - Random arrival and processing times
    - ▷ Jobs in manufacturing system
    - ▷ Customers in bank
  - In financial simulations, unknown future returns.
- Randomness in components driving simulation
  - Simulation output also random
  - **Example:** Repeat manufacturing simulation 6 times:

$$19.1, \quad 42.7, \quad 28.3, \quad 29.1, \quad 40.6, \quad 24.9$$

  - Must use statistical techniques to analyze results.

Introduction

- Intro statistics course typically assumes data
  - independent & identically distributed (IID)
  - normally distributed
- Flow times in manufacturing system
  - *Dependent*
    - ▷ If one job has unusually long wait
    - ▷ Then next job probably also will
  - *Not identically distributed*
    - ▷ First job has no wait; others may have to wait.
  - *Not normally distributed*
    - ▷ Always positive
    - ▷ Often skewed to right, with possible mode at zero

Introduction

- Thus, may not be able to analyze simulation output using classical statistical methods.
- In these notes, discuss statistical methods for
  - Designing simulation experiments
  - Analyzing simulation output
- **General problem:** compute **performance measure** α related to stochastic model being simulated.
- **Goals**
  - Develop **estimator** $\widehat{\alpha}_n$ of α based on "simulation effort" n.
  - Provide **measure of statistical error** of estimator $\widehat{\alpha}_n$.

Performance Measures

- How to measure how "good" system is?
  - expected number of jobs completed in first 4 hours of a day
  - long-run average flow time
- Mainly two types of performance measures:
  1. *Terminating* (AKA *transient* or *finite-horizon*)
     - Evolution over finite time horizon
     - Most finance problems are of this type
  2. *Steady-state* (AKA *long-run* or *infinite-horizon*)
     - Evolution over infinite time horizon

Terminating Simulation

- **Goal**: To calculate
$$\mu = \mathcal{E}[X]$$
  - X = random variable representing system performance over finite horizon
  - $\mathcal{E}[\,\cdot\,]$ denotes expectation or "average"
- "Initial conditions" $\mathcal{I}$
  - Conditions under which system starts.
  - Queueing simulations often start with no customers.
  - Large impact on performance measure since only finite horizon.

Terminating Simulation

Example: Manufacturing System

- Jobs accepted between 9:00am – 5:00pm
- Each day, system runs until all accepted parts are finished.
- $X =$ (random) number jobs completed in first 4 hours of day
- Terminating performance measure:
  - $\mathcal{E}[X] =$ Expected # jobs completed in first 4 hours of day
  - $\mathcal{P}\{X \geq 50\} = \mathcal{E}[Y]$, where $Y = I(X \geq 50)$
    - $I(A) = 1$ if event A occurs, $I(A) = 0$ otherwise.
- Initial conditions $\mathcal{I}$: system begins day empty
- Implicit assumption: each day has same dynamics.

Terminating Simulation

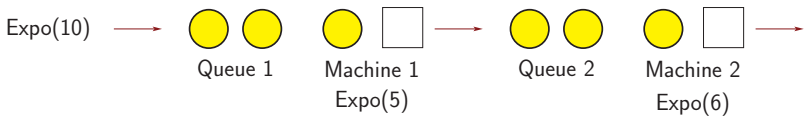
Example: Portfolio Valuation

- Given a portfolio of assets
  - e.g., 100 shares of GOOG, and 200 shares of MSFT
- $X_0 =$ current value of portfolio.
- $X =$ value of portfolio one year from now.
- Terminating performance measure:
  - $\mathcal{E}[X] =$ Expected value of portfolio one year from now
  - $\mathcal{P}\{X < 0.9X_0\} = \mathcal{E}[Y]$, where $Y = I(X < 0.9X_0)$
- Initial conditions $\mathcal{I}$: current value of portfolio X_0

Output Analysis for Terminating Simulations

- Basic approach to estimate $\mu = \mathcal{E}[X]$:
  1. Run simulation for n IID replications, generating
     $$X_1, X_2, \ldots, X_n$$
  2. Form *sample mean* $\bar{X}(n)$ as estimator for μ:
     $$\bar{X}(n) = \frac{1}{n}[X_1 + X_2 + \cdots + X_n] = \frac{1}{n}\sum_{i=1}^{n} X_i$$
- IID replications
  - *Identically distributed*: Run each replication with
    - same initial conditions $\mathcal{I}$
    - same system dynamics
  - *Independent*: Across replications, use different streams of random numbers.

Example: Manufacturing System



- $X = \#$ jobs completed in first 4 hours of day
- **Goal:** to estimate $\mu = \mathcal{E}[X]$
- Simulate 50 independent days, yielding
  $$19,\ 23,\ 18,\ \ldots,\ 28$$
- Estimate of μ is
  $$\bar{X}(50) = \frac{1}{50}[19 + 23 + 18 + \cdots + 28] = 22.5$$
- Is $\bar{X}(50) = 22.5$ a good estimate of μ?

Example: Portfolio Valuation

- Suppose $X_0 = \$40{,}000$, which is current value of portfolio
- X = value of portfolio one year from now
- **Goal:** to estimate $\mu = \mathcal{E}[X]$ and $p = \mathcal{E}[I(X < 0.9X_0)]$
- Simulate 500 independent replications, yielding outputs of X as

$$32691,\ 47593,\ 37567,\ \ldots,\ 28301$$

- For each X_i, compute $Y_i = I(X_i < 0.9X_0)$:

$$1,\ 0,\ 0,\ \ldots,\ 1$$

- Estimates of μ and p are

$$\bar{X}(500) = 47983, \qquad \hat{p}(500) = \bar{Y}(500) = 0.103$$

- Are these good estimates of μ and p?

Law of Large Numbers

- **Law of Large Numbers (LLN)**:
  If $X_1, X_2, \ldots$ are IID with $\mathcal{E}[|X_i|] < \infty$, then

$$\bar{X}(n) \;=\; \frac{1}{n}\sum_{i=1}^{n} X_i \;\approx\; \mu$$

  when sample size n is large.
  - Strong law of large numbers:

$$\mathcal{P}\left\{\lim_{n\to\infty} \bar{X}(n) = \mu\right\} \;=\; 1.$$

  - Weak law of large numbers: for every $\delta > 0$,

$$\lim_{n\to\infty} \mathcal{P}\left\{|\bar{X}(n) - \mu| < \delta\right\} \;=\; 1.$$

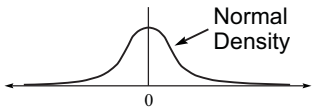
- So sample mean $\bar{X}(n)$ will be "close" to true mean μ when sample size is large.
- But how close?

Central Limit Theorem (CLT)

CLT: If $X_1, X_2, \ldots$ are IID with positive, finite variance σ^2, then

$$\frac{\sqrt{n}}{\sigma}\left(\bar{X}(n) - \mu\right) \;\overset{\mathcal{D}}{\approx}\; \mathcal{N}(0,1) \quad \text{for large } n$$

- $\overset{\mathcal{D}}{\approx}$ means "is approximately distributed as"
- $\mathcal{N}(0,1)$ is standard normal RV (mean 0, variance 1)



- More precisely, CLT states for every $x \in \mathbf{R}$,

$$\lim_{n\to\infty} \mathcal{P}\left\{\frac{\sqrt{n}}{\sigma}\left(\bar{X}(n) - \mu\right) \le x\right\} \;=\; \mathcal{P}\left\{\mathcal{N}(0,1) \le x\right\}$$

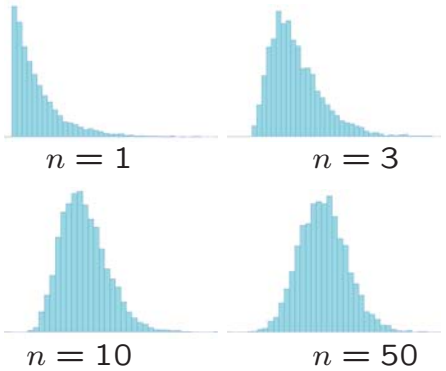
- Can rewrite CLT as

$$\sqrt{n}\left(\bar{X}(n) - \mu\right) \;\overset{\mathcal{D}}{\approx}\; \mathcal{N}(0,\sigma^2) \quad \text{or}$$

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \;\overset{\mathcal{D}}{\approx}\; \mathcal{N}(0,1) \quad \text{or } \ldots$$

Interpretation of CLT

- $X_1, X_2, \ldots, X_n$ IID exponential
- Histograms of $\frac{\sqrt{n}}{\sigma}\left(\bar{X}(n) - \mu\right)$, where $\sigma = \sqrt{\mathcal{V}(X_i)}$



Interpretation of CLT

- CLT: For large n,
$$\frac{\sqrt{n}}{\sigma}\left(\bar{X}(n)-\mu\right) \overset{\mathcal{D}}{\approx} \mathcal{N}(0,1)$$
- After a little algebra, can write estimator $\bar{X}(n)$ as
$$\bar{X}(n) \overset{\mathcal{D}}{\approx} \mu + \frac{\sigma}{\sqrt{n}}\mathcal{N}(0,1)$$
- Estimator's *error* $\overset{\mathcal{D}}{\approx} \frac{\sigma}{\sqrt{n}}\mathcal{N}(0,1)$
  - Depends on model simulated through *standard deviation* $\sigma = \sqrt{\mathcal{V}(X)} = \sqrt{\mathcal{E}[(X-\mu)^2]}$.
  - Depends on sample size n through $1/\sqrt{n}$.
  - Approximately normally distributed.

Scaling Factor in CLT

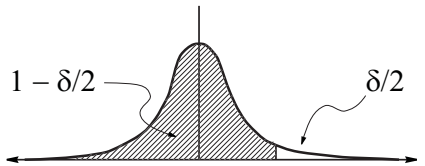
- **CLT:** $\frac{\sqrt{n}}{\sigma}\left(\bar{X}(n)-\mu\right) \overset{\mathcal{D}}{\approx} \mathcal{N}(0,1)$
- Why do we scale LHS by $\frac{\sqrt{n}}{\sigma}$?
- **Heuristic Argument**
  - Suppose instead scale LHS by constant c_n; i.e., LHS $= c_n\left(\bar{X}(n)-\mu\right)$.
  - Want LHS and RHS of CLT to have same variance (i.e., 1):
$$\begin{aligned}\mathcal{V}\left(c_n(\bar{X}(n)-\mu)\right) &= c_n^2\,\mathcal{V}\left(\bar{X}(n)-\mu\right) = c_n^2\,\mathcal{V}\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)\\ &= \frac{c_n^2}{n^2}\,\mathcal{V}\left(\sum_{i=1}^{n}X_i\right) = \frac{c_n^2}{n^2}\sum_{i=1}^{n}\mathcal{V}\left(X_i\right)\\ &= \frac{c_n^2\sigma^2}{n}.\end{aligned}$$
  - Choosing $c_n = \frac{\sqrt{n}}{\sigma}$ makes LHS have variance 1.

CLT With Estimated Variance

- **CLT:** $\frac{\sqrt{n}}{\sigma}\left(\bar{X}(n)-\mu\right) \overset{\mathcal{D}}{\approx} \mathcal{N}(0,1)$
- Typically, we do not know the value of σ.
- Estimate σ using sample standard deviation $S(n)$, where
$$S^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\bar{X}(n)\right)^2$$
- **Another CLT:** If $X_1, X_2, \ldots$ are IID with finite, positive variance σ^2, then for large n,
$$\frac{\sqrt{n}}{S(n)}\left(\bar{X}(n)-\mu\right) \overset{\mathcal{D}}{\approx} \mathcal{N}(0,1)$$
- Rule of Thumb
  - For CLT with σ, need $n \geq 30$.
  - For CLT with $S(n)$, need $n \geq 50$.

Output Analysis for Transient Simulations

- Now derive *confidence interval* (CI) for μ.
- First define *confidence level* $1-\delta$
  - Typically, $\delta = 0.05$ or 0.01.
- In z-table, look up constant $z \equiv z_{1-\delta/2}$:
$$\mathcal{P}\{\mathcal{N}(0,1) \leq z\} = 1-\delta/2$$
  - $z = 1.96$ for $\delta = 0.05$



- $\mathcal{P}\{-z \leq \mathcal{N}(0,1) \leq z\} = 1-\delta$ by symmetry.

Output Analysis for Transient Simulations

- CLT implies for large n,

$$1-\delta = \mathcal{P}\{\, -z \,\le\, \mathcal{N}(0,1) \,\le\, z \,\}$$

$$\approx \mathcal{P}\left\{ -z \,\le\, \frac{\sqrt{n}}{S(n)}(\bar{X}(n)-\mu) \,\le\, z \right\}$$

$$= \mathcal{P}\left\{ -\frac{zS(n)}{\sqrt{n}} \,\le\, \bar{X}(n)-\mu \,\le\, \frac{zS(n)}{\sqrt{n}} \right\}$$

$$= \mathcal{P}\left\{ \bar{X}(n)-\frac{zS(n)}{\sqrt{n}} \,\le\, \mu \,\le\, \bar{X}(n)+\frac{zS(n)}{\sqrt{n}} \right\}$$

- Approx $100(1-\delta)\%$ CI for true mean μ:

$$I_n \equiv \left[\, \bar{X}(n) \,\pm\, \frac{zS(n)}{\sqrt{n}} \,\right]$$

Output Analysis for Transient Simulations

Method to Construct CI for μ

1. Specify sample size $n \ge 50$ and
   z as $1-\delta/2$ critical pt of std normal.
2. Generate n IID replicates $X_1, X_2, \ldots, X_n$.
3. Calculate sample mean $\bar{X}(n)$ and sample variance $S^2(n)$:

$$\bar{X}(n) = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

$$S^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\bar{X}(n)\right)^2$$

4. Approx $100(1-\delta)\%$ CI for μ:

$$I_n \equiv \left[\, \bar{X}(n) \,\pm\, \frac{zS(n)}{\sqrt{n}} \,\right]$$

Transient Simulations: Arena

In Arena, you can run IID replications and construct CIs as follows:

1. Go to Run/Setup menu.
2. Click on "Replication Parameters" tab.
3. Specify the following parameters:
   - Number of Replications
   - Replication Length
4. In output report, the Half Width is for a 95% CI.

Example: Manufacturing System (cont.)

- **Goal:** estimate μ = expected # jobs completed in first 4 hours
- Simulate 50 independent days, giving $X_1, X_2, X_3, \ldots, X_{50}$ as

$$19,\ 23,\ 18,\ \ldots,\ 28$$

- Estimate of μ is $\bar{X}(50) = 22.5$
- Sample variance of X_i's is

$$S^2(50) = \frac{1}{49}\left[(19-22.5)^2+(23-22.5)^2+\cdots+(28-22.5)^2\right]$$

$$= 18.4$$

- Approximate 95% confidence interval for μ is

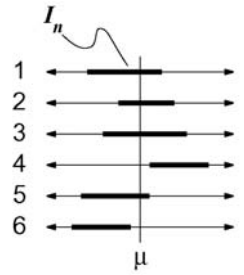
$$\left[\, 22.5 \,\pm\, \frac{1.96\times\sqrt{18.4}}{\sqrt{50}} \,\right] \quad = \quad [\,22.5 \,\pm\, 1.2\,]$$

Example: Portfolio Valuation (cont.)

- Recall: X_i = ending portfolio value on ith rep, and $Y_i = I(X_i < 0.9X_0)$
- From 500 replications, we get
  - $X_1, X_2, \ldots, X_{500}$ are 32691, 47593, 37567, ..., 28301
  - $Y_1, Y_2, \ldots, Y_{500}$ are 1, 0, 0, ..., 1
- Estimates of μ and p are
$$\bar{X}(500) = 47983, \qquad \hat{p}(500) = \bar{Y}(500) = 0.104$$
- Sample variances of X_i's and Y_i's are
$$S_x^2(500) = 1.03 \times 10^8, \qquad S_y^2(500) = 0.093$$
- Approximate 95% confidence interval for μ is
$$\left[47983 \pm \frac{1.96 \times \sqrt{1.03 \times 10^8}}{\sqrt{500}}\right] = [47983 \pm 888]$$
- Approximate 95% confidence interval for p is
$$\left[0.104 \pm \frac{1.96 \times \sqrt{0.093}}{\sqrt{500}}\right] = [0.104 \pm 0.027]$$

Output Analysis for Terminating Simulations

- Important to provide CI $I_n = [\bar{X}(n) \pm zS(n)/\sqrt{n}]$ for μ
  - Gives measure of error of estimator $\bar{X}(n)$
  - $\mathcal{P}(\mu \in I_n) \approx 1 - \delta$ for large n by CLT.
- If steps repeated m indep times, we get m different CIs.
  - Some CIs will contain (cover) μ, and others will not.
  - Approximately $(1 - \delta)m$ of m intervals should cover μ.



- **Definition:** *Coverage* of CI is actual prob that $\mu \in I_n$

Transient Simulations: Coverage Error

- Suppose that random variable X satisfies
  - $\mathcal{E}(X^4) < \infty$
  - X is a continuous random variable.
- Define
$$\hat{\sigma}^2(n) = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}(n)\right)^2$$
  - $\hat{\sigma}^2(n)$ is not the unbiased sample variance $S^2(n)$.
  - $\hat{\sigma}^2(n)$ is asymptotically unbiased, i.e.,
$$\mathcal{E}[\hat{\sigma}^2(n)] = \frac{n-1}{n}\sigma^2 \to \sigma^2 \text{ as } n \to \infty.$$
  - $\hat{\sigma}^2(n) \approx S^2(n)$ for large n,

Transient Simulations: Coverage Error

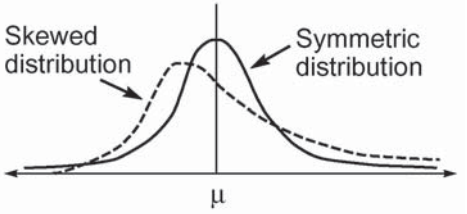
- Define
  - $F_n(x) = \mathcal{P}\left\{\frac{\sqrt{n}}{\tilde{\sigma}(n)}\left(\bar{X}(n) - \mu\right) \le x\right\}$
  - $\Phi(x) = \mathcal{P}\{\mathcal{N}(0,1) \le x\}$ is CDF of $\mathcal{N}(0,1)$
  - $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is density of $\mathcal{N}(0,1)$
- CLT: for every $x \in \mathbf{R}$,
$$\lim_{n\to\infty} F_n(x) = \Phi(x)$$
- **Edgeworth expansion** (CLT refinement):
$$F_n(x) = \Phi(x) + \frac{\mathcal{E}[(X-\mu)^3]}{6\sigma^3\sqrt{n}}(2x^2+1)\phi(x) + O\left(\frac{1}{n}\right),$$
where $O(1/n) \approx C/n$ for some constant C.

Transient Simulations: Coverage Error

- For large sample size n, error in CLT approximation (at x) is basically
$$\frac{\mathcal{E}[(X-\mu)^3]}{6\sigma^3\sqrt{n}}(2x^2+1)\phi(x)$$
  - Decreases at rate $1/\sqrt{n}$ in sample size n, which is quite slowly.
  - Depends on problem being simulated only through
$$\gamma \equiv \frac{\mathcal{E}[(X-\mu)^3]}{\sigma^3},$$
which is *skewness* of X.

Transient Simulations: Coverage Error

- Skewness of X is
$$\gamma = \frac{\mathcal{E}[(X-\mu)^3]}{\sigma^3}.$$
- If density of X is symmetric about mean, then skewness $\gamma = 0$.
- Skewness measures *asymmetry* of density of X.
- For the skewed distribution below, is $\gamma < 0$ or > 0?



Transient Simulations: Coverage Error

- The more symmetric the density of X is, then
  - better CLT approximation since smaller error term in Edgeworth expansion
  - more accurate approximation
$$\mathcal{P}\{\mu \in I_n\} \approx 1-\delta$$
- If X has highly asymmetric density (as is often the case in queueing and finance simulations),
  - CLT approximation not so good
  - Coverage of CI I_n may be significantly less than $1-\delta$.
  - e.g., nominal 95% CI may have coverage of only 0.7 for small sample sizes n.
  - Need very large n for CI to have good coverage.

Transient Simulations: Pre-specifying CI Widths

- Confidence Interval (CI):
$$\left[\ \bar{X}(n) \pm \frac{z\,S(n)}{\sqrt{n}}\ \right]$$
has half-width
$$H_n \equiv \frac{z\,S(n)}{\sqrt{n}}$$
- H_n is random
  - Typically don't know H_n before running simulation.
- Suppose we want CI to have half-width ϵ; i.e., want CI
$$\left[\ \bar{X}(n) \pm \epsilon\ \right]$$
- How many replicates n should we generate?

Terminating Simulations: Pre-specifying CI Widths

- Set $H_n = \epsilon$ and solve for n:

$$\frac{z\,S(n)}{\sqrt{n}} = \epsilon \qquad \Rightarrow \qquad n = \frac{z^2\,S^2(n)}{\epsilon^2}$$

- $S(n) \approx \sigma$ for large n
- **Two-stage procedure**
  - *First Stage*: Simulate n *pilot runs* and compute $S^2(n)$
  - *Second Stage*: Sample size N for *production runs* is

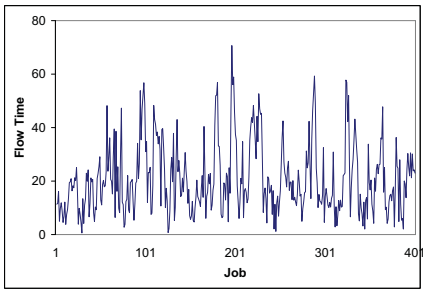
$$N = \frac{z^2\,S^2(n)}{\epsilon^2}$$

- **Example:** If $S^2(50) = 18.4$ and want half-width $\epsilon = 0.5$, then

$$N = \frac{(1.96)^2 \times 18.4}{(0.5)^2} \approx 283$$

Steady-State Performance Measures

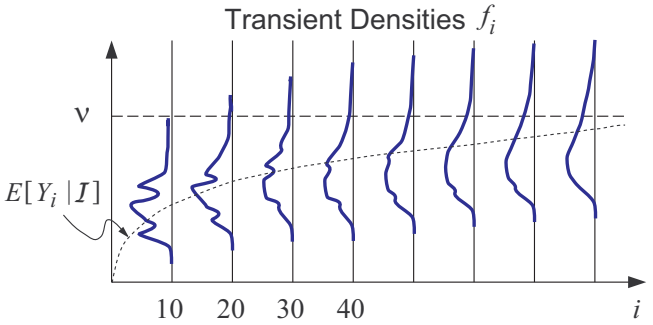
- System runs 24 hours/day.
- Y_i = flow time of ith part when system runs 24 hrs/day



  - Flow times are **dependent**
  - Flow times of early jobs are smaller and less variable
  - Eventually reach "equilibrium"

Steady-State Simulations

- Assume $(Y_1, Y_2, Y_3, \ldots)$ "settles down" to steady state Y_∞



For large i,
$Y_i \overset{\mathcal{D}}{\approx} Y_\infty$ and
$f_i \approx f_\infty$

- **Example:** Y_i = flow time of ith job.
- **Goal:** to produce confidence intervals for $\nu = \mathcal{E}[Y_\infty]$, where

$$\frac{1}{m}\,[\,Y_1 + Y_2 + \cdots + Y_m\,] \approx \nu, \quad \text{for large } m$$

Steady-State Performance Measures

- **Example:** Manufacturing system operates 24 hrs/day
- Y_i = flow time of ith job
- Assume system eventually "settles down" into steady state Y_∞.
- Steady-state performance measure:
  - $\mathcal{E}[Y_\infty]$ = expected steady-state flow time of jobs
  - Steady-state prob that flow time is at most 20:

$$\mathcal{P}\{Y_\infty \le 20\} = \mathcal{E}[I(Y_\infty \le 20)]$$

- Initial conditions $\mathcal{I}$:
  - No customers present at beginning of first day.
  - Over time, effects of initial conditions "wash away."

Steady-State Performance Measures

Many systems do not have steady state

- Manufacturing system operates only from 9am–5pm.
- Y_i = flow time of ith job to arrive since first day of operation.
- Process $\mathbf{Y} = (Y_1, Y_2, Y_3, \ldots)$ does not have steady state
  - First job of each day always has no wait
  - Other jobs may have to wait
- If system operates 24 hrs/day, then steady state *may* exist.
  - May not if system dynamics change over time.
    - ▷ e.g., more jobs on Friday than Tuesday

Steady-State Cycle Measures

Even if process $\mathbf{Y} = (Y_1, Y_2, Y_3, \ldots)$ does not have steady state, can sometimes instead look at **cycles**.

- Manufacturing system operates 24 hours/day.
- Arrival rate higher during day than night with daily cycle.
- Y_i = flow time of ith job to arrive since first day of operation.
- Process $\mathbf{Y} = (Y_1, Y_2, Y_3, \ldots)$ does not have steady state
- Instead let C_j = average flow time of all jobs on the jth day.
- Process $\mathbf{C} = (C_1, C_2, C_3, \ldots)$ may have steady state
  - May not if system dynamics are different on different days.
    - ▷ e.g., more jobs on Friday than Tuesday

Steady-State Simulations

- Assume process eventually "settles down" to steady state Y_∞.
  1. Discrete-time process:
     - $\mathbf{Y} = (Y_i : i = 1, 2, \ldots)$ has integer-valued time index
     - **Goal:** to produce confidence intervals for $\nu = \mathcal{E}[Y_\infty]$, where
     $$\frac{1}{m}\sum_{i=1}^{m} Y_i \;\to\; \nu, \quad \text{as} \;\; m \to \infty.$$
  2. Continuous-time process:
     - $\mathbf{Y} = (Y(t) : t \geq 0)$ has continuous-valued time index
     - **Goal:** to produce confidence intervals for $\nu = \mathcal{E}[Y_\infty]$, where
     $$\frac{1}{t}\int_0^t Y(s)\, ds \;\to\; \nu, \quad \text{as} \;\; t \to \infty.$$

Steady-State Simulations: Examples

- Discrete-time process $\mathbf{Y} = (Y_i : i = 1, 2, \ldots)$
  - Y_i = flow time of ith job in queueing system
  - ν = steady-state expected flow time.
- Continuous-time process $\mathbf{Y} = (Y(t) : t \geq 0)$
  - $Y(t)$ = number of jobs in system at time t.
  - ν = long-run time-average number of jobs in system.
- We'll only cover discrete-time process
  - Continuous-time process can be handled similarly.
  - **Key issue:** $Y_1, Y_2, Y_3, \ldots$ are **dependent**.

Steady-State Simulations

- **Goal:** to estimate steady-state mean ν of discrete-time process $(Y_i : i = 1, 2, \ldots)$ for which steady state exists.
- **Point estimator** for ν:
$$\bar{Y}(m) \equiv \frac{1}{m}\sum_{i=1}^{m} Y_i$$
- $Y_1, Y_2, \ldots$ **not IID**, but $\bar{Y}(m)$ still satisfies a LLN and CLT, which differ from ordinary LLN and CLT from before.
- **LLN**: $\bar{Y}(m) \approx \nu$ for large m.
- **CLT**: $\exists$ constant $\psi > 0$ such that for very large m,
$$\frac{\sqrt{m}}{\psi}\left[\bar{Y}(m) - \nu\right] \overset{\mathcal{D}}{\approx} \mathcal{N}(0, 1).$$
- **Definition:** ψ^2 is *time-average variance constant* (TAVC):
$$\psi^2 = \lim_{m\to\infty} \mathcal{V}[\sqrt{m}(\bar{Y}(m) - \nu)] = \lim_{m\to\infty} m\,\mathcal{V}[\bar{Y}(m)]$$

Steady-State Simulations: Difficulties

- Unfortunately, difficult to estimate TAVC
$$\psi^2 = \lim_{m\to\infty} m\,\mathcal{V}[\bar{Y}(m)], \quad \text{where } \bar{Y}(m) = \frac{1}{m}\sum_{i=1}^{m} Y_i$$
- If $(Y_i : i = 1, 2, \ldots)$ is stationary process,
$$\psi^2 = \mathcal{V}[Y_1] + 2\sum_{k=1}^{\infty} \mathrm{Cov}[Y_1, Y_{1+k}]$$
- Sample variance $S_Y^2(m)$ does **not** estimate ψ^2:
$$\begin{aligned} S_Y^2(m) &= \frac{1}{m-1}\sum_{i=1}^{m}[Y_i - \bar{Y}(m)]^2 \\ &= \left(\frac{1}{m-1}\sum_{i=1}^{m} Y_i^2\right) - \frac{m}{m-1}\bar{Y}(m)^2 \end{aligned}$$
instead estimates $\mathcal{V}[Y_1] = \mathcal{E}[Y_1^2] - \nu^2$.
  - $S_Y^2(m)$ **doesn't capture sum of covariances** $2\sum_{k=1}^{\infty} \mathrm{Cov}[Y_1, Y_{1+k}] \neq 0$ for **correlated** $Y_1, Y_2, \ldots$.
- **One solution**: *Method of Multiple Replications*

Steady-State Simulations: Method of Multiple Replications

- Simulate r *IID* replications, each of (large) length k
  - *Identically Distributed*: In each replication, use
    - ▷ Same initial conditions $\mathcal{I}$
    - ▷ Same system dynamics
  - *Independent*: Across replications,
    - ▷ Use non-overlapping streams of random numbers.
- Compute sample average within each replication.
- This yields r IID outputs.
- Form sample variance across reps to construct CI.

Steady-State Simulations: Method of Multiple Replications

- Run r IID replications, each with k outputs, with k large

	Outputs					Sample Mean Within Replication
replication 1	$Y_{1,1}$	$Y_{1,2}$	$Y_{1,3}$	$\cdots$	$Y_{1,k}$	W_1
replication 2	$Y_{2,1}$	$Y_{2,2}$	$Y_{2,3}$	$\cdots$	$Y_{2,k}$	W_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
replication r	$Y_{r,1}$	$Y_{r,2}$	$Y_{r,3}$	$\cdots$	$Y_{r,k}$	W_r

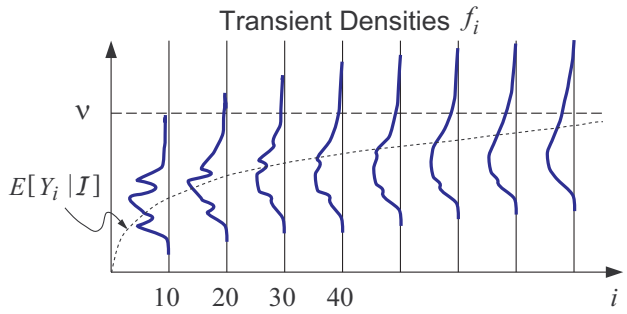
- $W_j = \frac{1}{k}\left[Y_{j,1} + Y_{j,2} + \cdots + Y_{j,k}\right]$ is sample mean of jth rep
  - CLT: $\frac{\sqrt{k}}{\psi}[W_j - \nu] \overset{\mathcal{D}}{\approx} \mathcal{N}(0, 1)$, so $W_j \overset{\mathcal{D}}{\approx} \mathcal{N}(\nu, \psi^2/k)$.
  - $W_1, W_2, \ldots, W_r$ are **IID** and **roughly normal**.
- Compute sample mean $\overline{W}(r) = \frac{1}{r}\sum_{j=1}^{r} W_j$ and sample var $S_W^2(r) = \frac{1}{r-1}\sum_{j=1}^{r}[W_j - \overline{W}(r)]^2$ across reps.

Steady-State Simulations: Method of Multiple Replications

- Since $W_1, W_2, \ldots, W_r$ are IID and approx $\mathcal{N}(\nu, \psi^2/k)$,
$$\frac{\overline{W}(r) - \nu}{\sqrt{S_W^2(r)/r}} \overset{\mathcal{D}}{\approx} T_{r-1} \quad \text{(Student } t \text{ with } r-1 \text{ d.f.)}$$
- Approx $100(1-\delta)\%$ CI for ν:
$$\left[\overline{W}(r) \pm \frac{t_{1-\delta/2}\, S_W(r)}{\sqrt{r}} \right]$$
  - $t_{1-\delta/2} = (1-\delta/2)$-critical point of T_{r-1} (from t-table)
  - Use Student t critical point instead of normal because number r of reps often small so that run length k within rep is large.
- **Problem:** estimator $\overline{W}(r)$ significantly contaminated by **initialization bias**.

Steady-State Simulations: Method of Multiple Replications

- **Problem:** Each replication starts with initial condition $\mathcal{I}$ atypical of steady state
  - Queueing simulations often start with system empty



For large i, $Y_i \overset{\mathcal{D}}{\approx} Y_\infty$ and $f_i \approx f_\infty$

- **Solution:** *Initial-data deletion*

Steady-State Simulations: Initial-Data Deletion

- Suppose first c outputs in replication are "contaminated." Rest are "clean."

	"contaminated"			"clean"			Sample Mean of Clean Data
rep 1	$Y_{1,1}$	$\cdots$	$Y_{1,c}$	$Y_{1,c+1}$	$\cdots$	$Y_{1,k}$	U_1
rep 2	$Y_{2,1}$	$\cdots$	$Y_{2,c}$	$Y_{2,c+1}$	$\cdots$	$Y_{2,k}$	U_2
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
rep r	$Y_{r,1}$	$\cdots$	$Y_{r,c}$	$Y_{r,c+1}$	$\cdots$	$Y_{r,k}$	U_r

- Compute sample mean of only $k-c$ "clean" data in each rep j
$$U_j = \frac{1}{k-c}\left[Y_{j,c+1} + Y_{j,c+2} + \cdots + Y_{j,k} \right],$$
- CLT: $\frac{\sqrt{k-c}}{\psi}[U_j - \nu] \overset{\mathcal{D}}{\approx} \mathcal{N}(0,1)$, so $U_j \overset{\mathcal{D}}{\approx} \mathcal{N}(\nu, \psi^2/(k-c))$.
- Compute sample mean $\bar{U}(r) = \frac{1}{r}\sum_{j=1}^r U_j$ and sample variance $S_U^2(r) = \frac{1}{r-1}\sum_{j=1}^r [U_j - \bar{U}(r)]^2$ of IID $U_1, U_2, \ldots, U_r$.

Steady-State Simulations: Initial-Data Deletion

- Since $U_1, U_2, \ldots, U_r$ are IID and approx $\mathcal{N}(\nu, \psi^2/(k-c))$,
$$\frac{\bar{U}(r) - \nu}{\sqrt{S_U^2(r)/r}} \overset{\mathcal{D}}{\approx} T_{r-1} \quad \text{(Student } t \text{ with } r-1 \text{ d.f.)}$$
- Approximate $100(1-\delta)\%$ CI for ν is
$$\left[\bar{U}(r) \pm \frac{t_{1-\delta/2}\, S_U(r)}{\sqrt{r}} \right]$$

Remarks

- Problem: delete c outputs from each of r replications.
  - Deleting total of rc outputs over all replications.
- Methods that run only one replication discard only c data points.
- Many heuristics to determine number data points to discard, e.g., see Law (2014), Section 9.5.2.

Steady-State Simulations: Initial-Data Deletion

In Arena, you can implement method of multiple replications with initial-data deletion as follows:

1. Go to Run/Setup menu.
2. Click on "Replication Parameters" tab.
3. Specify the following parameters:
   - Number of Replications
   - Warm-up Period
   - Replication Length

Steady-State Simulations: Single Replication

- **Goal:** to estimate the steady-state mean $\nu = \mathcal{E}[Y_\infty]$ of discrete-time process $(Y_i : i = 1, 2, \ldots)$
- Before, we discussed the method of multiple replications to attack this problem.
- We now will examine a technique that uses only one replication.
- **Basic Idea:** Typically in practice, Y_i and Y_{i+k} are almost independent when k is **large**.
  - For example, suppose Y_i is flow time of the ith job in a queueing system.
  - Expect that flow time of 400th customer is almost independent of 300th job's flow time.

Steady-State Simulations: Single Replication

- Now suppose that we run a single simulation of length m.
- This gives us outputs $Y_1, Y_2, \ldots, Y_m$.
- Suppose we group the m outputs into $b \geq 2$ **large batches**, where
  - First batch consists of first $k = m/b$ outputs
  - Second batch consists of next k outputs, and so on.

Batch 1	Batch 2	$\cdots$	Batch b
$\underbrace{Y_1 \cdots Y_k}_{\text{first } k}$	$\underbrace{Y_{k+1} \cdots Y_{2k}}_{\text{next } k}$	$\cdots$	$\underbrace{Y_{(b-1)k+1} \cdots Y_{bk}}_{\text{last } k}$

Steady-State Simulations: Single Replication

- If batch size k is chosen to be large, then
  - Most of the outputs in one batch will be almost independent of most of the outputs in any other batch.
  - The only dependence that essentially exists is between outputs in two adjacent batches.
  - Outputs in batches that are not adjacent are almost independent.
  - If we compute the sample mean of each batch, then the b batch means should be almost independent.
  - Each batch mean is close to normally distributed for large k, since it is a sample mean and so a corresponding CLT holds.
- So the b **batch means** should be **roughly IID normal**.

Steady-State Simulations: Method of Batch Means

1. Select a total run length m, which is **very large**.
   - "Very large" depends on context, but in general, should choose $m \geq 1000$.
   - Also, select b as the number of batches.
   - In practice, should choose $10 \leq b \leq 30$, depending on m.
     - If m is "small," then you should choose b to be 10.
     - If m is "large," then b can be larger.

2. Run a simulation generating a total of m outputs.
   - This results in outputs $Y_1, Y_2, \ldots, Y_m$.

3. Group the m outputs into b batches, each of size $k = m/b$.

4. For each batch $j = 1, 2, \ldots, b$, compute batch mean

$$\overline{Y}_j(k) \;=\; \frac{1}{k} \sum_{\ell=1}^{k} Y_{(j-1)k+\ell}$$

Batch 1	Batch 2	$\cdots$	Batch b
$\underbrace{Y_1 \;\cdots\; Y_k}_{\overline{Y}_1(k)}$	$\underbrace{Y_{k+1} \;\cdots\; Y_{2k}}_{\overline{Y}_2(k)}$	$\cdots$	$\underbrace{Y_{(b-1)k+1} \;\cdots\; Y_{bk}}_{\overline{Y}_b(k)}$

CLT: $\frac{\sqrt{k}}{\psi}[\overline{Y}_j(k) - \nu] \overset{\mathcal{D}}{\approx} \mathcal{N}(0,1)$ for large k, so each $\overline{Y}_j(k) \overset{\mathcal{D}}{\approx} \mathcal{N}(\nu, \psi^2/k)$.

5. Since $\overline{Y}_1(k), \overline{Y}_2(k), \ldots, \overline{Y}_b(k)$ approx IID $\mathcal{N}(\nu, \psi^2/k)$,

$$\frac{\widehat{\nu}(b,k) - \nu}{\sqrt{S^2_{\overline{Y}}(b,k)/b}} \overset{\mathcal{D}}{\approx} T_{b-1} \quad \text{(Student } t \text{ with } b-1 \text{ d.f.)}$$

   - Point estimate for ν is

$$\widehat{\nu}(b,k) \;=\; \frac{1}{b} \sum_{j=1}^{b} \overline{Y}_j(k)$$

   - Sample variance of b batch means is

$$S^2_{\overline{Y}}(b,k) \;=\; \frac{1}{b-1} \sum_{j=1}^{b} \left[\overline{Y}_j(k) - \widehat{\nu}(b,k)\right]^2$$

   - Approximate $100(1-\delta)\%$ CI for ν is

$$\left[\; \widehat{\nu}(b,k) \;\pm\; \frac{t_{1-\delta/2}\, S_{\overline{Y}}(b,k)}{\sqrt{b}} \;\right]$$

   where $t_{1-\delta/2}$ is $(1-\delta/2)$-critical point of T_{b-1}.

Remarks on Batch Means (BM)

- **Key Assumption:** Run length is "very long" so that batch means are roughly normally distributed and independent.
  - Highly correlated $Y_1, Y_2, \ldots$ requires longer run lengths.
- Arena uses BM to build CIs when "Number of Replications" $= 1$
  - Unspecified algorithm to determine number b of batches.
  - If Replication Length not long enough, then Arena output reports "insufficient" or "correlated."
- Can modify our BM procedure to use **initial-data deletion**.
  - In Arena, specify "Warm-up Period".
  - Multiple replications: discard initial data **in each rep**.
  - Batch means: Delete initial data **only once**.
- **Rule of thumb:** Single-replicate procedures yield steady-state estimators with smaller MSE than multiple-replicate ones.

Steady-State Simulations: Other Methods

- Spectral methods
  - Anderson, T. W. 1994. *The Statistical Analysis of Time Series*. Wiley.
- Regenerative methods
  - Crane, M. and D. L. Iglehart. 1975. Simulating stable stochastic systems, III: Regenerative processes and discrete-event simulations. *Operations Research*, 23:33–45.
  - Shedler, G. S. 1993. *Regenerative Stochastic Simulation*. Academic Press.
- Standardized time series
  - Schruben, L. W. 1983. Confidence interval estimation using standardized time series. *Operations Research*, 31:1090–1108.
- Two-stage procedures
  - Nakayama, M. K. 1994. Two-Stage Stopping Procedures Based on Standardized Time Series. *Management Science*, 40:1189–1206.

Confidence Intervals for Other Parameters

- Suppose α is parameter related to stoch. model being simulated.
- General approach to develop CI for α
  1. Develop **estimator** $\widehat{\alpha}_n$ of α based on "simulation effort" n.
  2. Show $\widehat{\alpha}_n$ satisfies **LLN**: $\widehat{\alpha}_n \approx \alpha$ for large n.
  3. Show $\widehat{\alpha}_n$ satisfies **CLT**: $\exists$ constant $0 < \tau < \infty$ s.t.
     $$\sqrt{n}(\widehat{\alpha}_n - \alpha) \stackrel{\mathcal{D}}{\approx} \mathcal{N}(0, \tau^2) \text{ for large } n.$$
  4. Develop **consistent estimator** $\widehat{\tau}_n$ of τ, i.e., $\widehat{\tau}_n \approx \tau$ for large n.
  5. 95% CI: $\left[\widehat{\alpha}_n \pm \frac{1.96\widehat{\tau}_n}{\sqrt{n}}\right]$
- Often interested in α that is not a mean or steady-state mean.
  - e.g., $\alpha = \frac{\mu_1}{\mu_2}$, or $\alpha =$ quantile (value-at-risk), or ...
  - Corresponding estimator $\widehat{\alpha}_n$ may not be a sample average.

Confidence Intervals for Other Parameters

- **Problem:** How to develop LLN and CLT for estimator $\widehat{\alpha}_n$ that is **not** a sample average?
- One approach:
  1. Show (e.g., via Taylor approximation) that for large n,
     $$\widehat{\alpha}_n - \alpha \approx \lambda(\bar{Z}_n - \mathcal{E}[Z])$$
     - $\bar{Z}_n$ is sample average of n IID copies of some RV Z
     - λ is some constant
  2. $\bar{Z}_n$ satisfies LLN, so $\bar{Z}_n - \mathcal{E}[Z] \approx 0$, giving LLN: $\widehat{\alpha}_n \approx \alpha$.
  3. $\bar{Z}_n$ satisfies CLT when $0 < \mathcal{V}[Z] < \infty$:
     $$\sqrt{n}(\bar{Z}_n - \mathcal{E}[Z]) \stackrel{\mathcal{D}}{\approx} \mathcal{N}(0, \mathcal{V}[Z]),$$
     so $\widehat{\alpha}_n$ satisfies CLT
     $$\sqrt{n}(\widehat{\alpha}_n - \alpha) \stackrel{\mathcal{D}}{\approx} \mathcal{N}(0, \lambda^2\mathcal{V}[Z]).$$

Nonlinear Functions of Means

- **Goal:** Given "smooth" nonlinear function g, we want to estimate
  $$\alpha = g(\mu_1, \mu_2, \ldots, \mu_d)$$
  - $\mu_i = \mathcal{E}[X^{(i)}]$, $i = 1, 2, \ldots, d$
- To keep notation simple, we focus on case $d = 2$, so
  $$\alpha = g(\mu_x, \mu_y)$$
  - $\mu_x = \mathcal{E}[X]$ and $\mu_y = \mathcal{E}[Y]$
  - X and Y may be dependent
- **Example:** In combat simulation, might want to estimate
  $$\alpha = \frac{\mathcal{E}[R]}{\mathcal{E}[B]} = g(\mathcal{E}[R], \mathcal{E}[B])$$
  where R and B are numbers of red and blue losses, so $g(x, y) = x/y$.

Nonlinear Functions of Means

- **Goal:** given smooth function $g(x, y)$, want to estimate
$$\alpha \;=\; g(\mu_x, \mu_y)$$
  - $\mu_x = \mathcal{E}[X]$ and $\mu_y = \mathcal{E}[Y]$
  - X and Y may be dependent
- Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be IID copies of (X, Y), and let
$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$
- $\bar{X}_n \approx \mu_x$ and $\bar{Y}_n \approx \mu_y$ by LLN.
- An estimator of $\alpha = g(\mu_x, \mu_y)$ is then
$$\widehat{\alpha}_n \;=\; g(\bar{X}_n, \bar{Y}_n) \;\approx\; g(\mu_x, \mu_y)$$
since $g(x, y)$ is smooth function.
- How do we construct CI for α?

Nonlinear Functions of Means

- Recall: estimator $\widehat{\alpha}_n \;=\; g(\bar{X}_n, \bar{Y}_n) \;\approx\; g(\mu_x, \mu_y) \;=\; \alpha$
- Define
  - $g_x = \left[\frac{\partial}{\partial x} g(x, y)\right]_{(x,y)=(\mu_x,\mu_y)}$,
    $g_y = \left[\frac{\partial}{\partial y} g(x, y)\right]_{(x,y)=(\mu_x,\mu_y)}$
  - $Z_i \;=\; g_x \cdot [X_i - \mu_x] \;+\; g_y \cdot [Y_i - \mu_y]$
  - $\bar{Z}_n = (1/n)\,\Sigma_{i=1}^{n} Z_i$
- **Delta Method:** Taylor's series approximation gives
$$\begin{aligned}\widehat{\alpha}_n - \alpha \;&=\; g(\bar{X}_n, \bar{Y}_n) \;-\; g(\mu_x, \mu_y)\\ &\approx\; g_x \cdot \left[\bar{X}_n - \mu_x\right] \;+\; g_y \cdot \left[\bar{Y}_n - \mu_y\right]\\ &=\; \frac{1}{n}\sum_{i=1}^{n} \left(g_x \cdot [X_i - \mu_x] \;+\; g_y \cdot [Y_i - \mu_y]\right)\\ &=\; \bar{Z}_n\end{aligned}$$

Nonlinear Functions of Means

- Recall $Z_i \;=\; g_x \cdot [X_i - \mu_x] \;+\; g_y \cdot [Y_i - \mu_y]$
  - $g_x = \left[\frac{\partial}{\partial x} g(x, y)\right]_{(x,y)=(\mu_x,\mu_y)}$ and $g_y = \left[\frac{\partial}{\partial y} g(x, y)\right]_{(x,y)=(\mu_x,\mu_y)}$
- $Z_1, \ldots, Z_n$ are IID since $(X_1, Y_1), \ldots, (X_n, Y_n)$ are IID
- $\mathcal{E}[Z_i] = 0$ since
$$\mathcal{E}[Z_i] \;=\; g_x \cdot \mathcal{E}[X_i - \mu_x] \;+\; g_y \cdot \mathcal{E}[Y_i - \mu_y]$$
- Since $\widehat{\alpha}_n - \alpha \approx \bar{Z}_n = \bar{Z}_n - \mathcal{E}[Z]$, CLT implies for large n,
$$\sqrt{n}\,[\widehat{\alpha}_n - \alpha] \;\approx\; \sqrt{n}\left(\bar{Z}_n - \mathcal{E}[Z]\right) \;\overset{\mathcal{D}}{\approx}\; \mathcal{N}(0, \sigma_z^2)$$
  - $\sigma_z^2 \;\equiv\; \mathcal{V}[Z] \;=\; g_x^2\,\mathcal{V}[X] + g_y^2\,\mathcal{V}[Y] + 2\,g_x\,g_y\,\mathrm{Cov}[X, Y]$
  - Approx $100(1-\delta)\%$ CI for α is
$$\left[\;\widehat{\alpha}_n \;\pm\; \frac{z_{1-\delta/2}\,\sigma_z}{\sqrt{n}}\;\right]$$

Nonlinear Functions of Means

- Recall
$$\sigma_z^2 \;=\; \mathcal{V}[Z] \;=\; g_x^2\,\mathcal{V}[X] \;+\; g_y^2\,\mathcal{V}[Y] \;+\; 2\,g_x\,g_y\,\mathrm{Cov}[X, Y]$$
  - $g_x = \left[\frac{\partial}{\partial x} g(x, y)\right]_{(x,y)=(\mu_x,\mu_y)}$ and $g_y = \left[\frac{\partial}{\partial y} g(x, y)\right]_{(x,y)=(\mu_x,\mu_y)}$
- We can estimate σ_z^2 by
$$S_z^2 \;=\; \widehat{g}_x^2\,S_x^2 \;+\; \widehat{g}_y^2\,S_y^2 \;+\; 2\,\widehat{g}_x\,\widehat{g}_y\,C_n$$
  - $\widehat{g}_x = \frac{\partial}{\partial x} g(\bar{X}_n, \bar{Y}_n)$ and $\widehat{g}_y = \frac{\partial}{\partial y} g(\bar{X}_n, \bar{Y}_n)$
  - $S_x^2 = \frac{1}{n-1}\Sigma_{i=1}^{n}(X_i - \bar{X}_n)^2$
  - $S_y^2 = \frac{1}{n-1}\Sigma_{i=1}^{n}(Y_i - \bar{Y}_n)^2$
  - $C_n = \frac{1}{n-1}\Sigma_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)$ is estimate of $\mathrm{Cov}[X, Y]$

Nonlinear Functions of Means

Algorithm: to construct CI for $\alpha = g(\mu_x, \mu_y)$

1. Generate n IID replicates $(X_1, Y_1), \ldots, (X_n, Y_n)$ of (X, Y).
2. Compute
$$\widehat{\alpha}_n \;=\; g(\bar{X}_n, \bar{Y}_n)$$
3. Approx $100(1-\delta)\%$ CI for α is
$$\left[\; \widehat{\alpha}_n \;\pm\; \frac{z_{1-\delta/2}\, S_z}{\sqrt{n}} \;\right]$$
where
$$S_z \;=\; \left(\widehat{g}_x^2\, S_x^2 \;+\; \widehat{g}_y^2\, S_y^2 \;+\; 2\,\widehat{g}_x\,\widehat{g}_y\, C_n\right)^{1/2}$$
and $\widehat{g}_x$, $\widehat{g}_y$, S_x^2, S_y^2, and C_n are defined on the previous slide.

Example: Ratio of Means

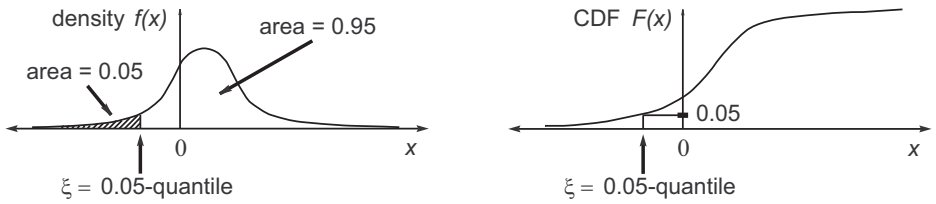
- Suppose we want to estimate $\alpha \;=\; \mathcal{E}[X]/\mathcal{E}[Y] \;=\; \mu_x/\mu_y$
- Then $g(x, y) = x/y$, so
$$\begin{aligned}
g_x \;&=\; \left[\frac{\partial}{\partial x}\, g(x,y)\right]_{(x,y)=(\mu_x,\mu_y)} \;=\; \frac{1}{\mu_y}, \\
g_y \;&=\; \left[\frac{\partial}{\partial y}\, g(x,y)\right]_{(x,y)=(\mu_x,\mu_y)} \;=\; \frac{-\mu_x}{\mu_y^2}, \\
\widehat{\alpha}_n \;&=\; g(\bar{X}_n, \bar{Y}_n) \;=\; \frac{\bar{X}_n}{\bar{Y}_n}, \\
\widehat{g}_x \;&=\; \frac{\partial}{\partial x}\, g(\bar{X}_n, \bar{Y}_n) \;=\; \frac{1}{\bar{Y}_n}, \\
\widehat{g}_y \;&=\; \frac{\partial}{\partial y}\, g(\bar{X}_n, \bar{Y}_n) \;=\; \frac{-\bar{X}_n}{\bar{Y}_n^2}, \\
S_z^2 \;&=\; \widehat{g}_x^2\, S_x^2 + \widehat{g}_y^2\, S_y^2 + 2\,\widehat{g}_x\,\widehat{g}_y\, C_n \;=\; \frac{1}{\bar{Y}_n^2}\, S_x^2 \;+\; \frac{\bar{X}_n^2}{\bar{Y}_n^4}\, S_y^2 \;-\; 2\,\frac{\bar{X}_n}{\bar{Y}_n^3}\, C_n \\
&=\; \frac{1}{\bar{Y}_n^2}\left[S_x^2 \;+\; \widehat{\alpha}_n^2\, S_y^2 \;-\; 2\,\widehat{\alpha}_n\, C_n\right]
\end{aligned}$$

Example: Ratio of Means

- Recall, our estimator of $\alpha = \mu_x/\mu_y$ is
$$\widehat{\alpha}_n = \frac{\bar{X}_n}{\bar{Y}_n}$$
- In general,
$$\mathcal{E}[\widehat{\alpha}_n] \;=\; \mathcal{E}\left[\frac{\bar{X}_n}{\bar{Y}_n}\right] \;\neq\; \frac{\mathcal{E}[\bar{X}_n]}{\mathcal{E}[\bar{Y}_n]} \;=\; \frac{\mu_x}{\mu_y}.$$
- Thus, $\widehat{\alpha}_n$ is typically a **biased** estimator of α.
- More generally, if g is a nonlinear function and $\alpha = g(\mu_x, \mu_y)$, then $\widehat{\alpha}_n = g(\bar{X}_n, \bar{Y}_n)$ is a biased estimator of α
- There are ways of reducing the bias, e.g., the jackknife.

Quantile Estimation: Example

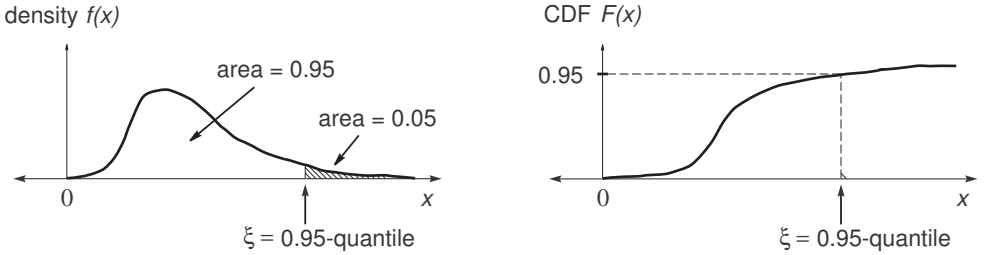
- Let X be RV denoting tomorrow's return on a portfolio.
- One measure of portfolio risk is the **value at risk (VaR)**.
- **Definition:** For a fixed probability p (say 0.05), the **VaR** is the constant ξ such that $\mathcal{P}(X \leq \xi) = p$



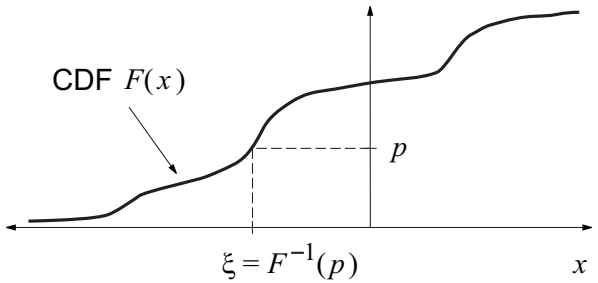
- $100p\%$ of days, portfolio will have return worse than ξ.

Quantile Estimation: Example

- Let $X \sim F$ be RV denoting the time to complete a project.
- Let ξ be the 0.95-quantile of distn of X
  - 95% chance of completing project by ξ, i.e., $P(X \leq \xi) = 0.95$



Quantile Estimation



Definition: For RV $X \sim$ CDF F and fixed $0 < p < 1$, p-**quantile** is constant ξ such that $F(\xi) \equiv \mathcal{P}(X \leq \xi) = p$,

$$\xi \;=\; F^{-1}(p) \;=\; \min\{x : F(x) \geq p\}.$$

- **Median** is the 0.5-quantile.
- **Inner quartile range** is (0.75-quantile) − (0.25-quantile)
- p-quantile is also known as $100p$-th **percentile**.

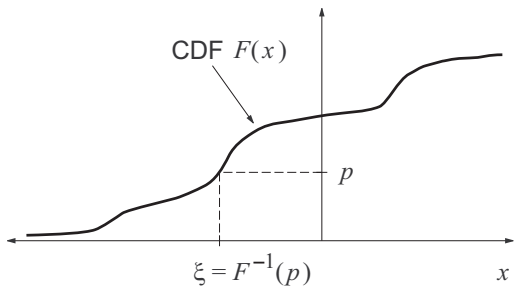
Quantile Estimation

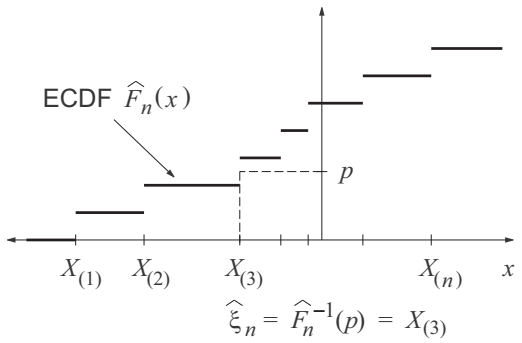
- How can we estimate $\xi = F^{-1}(p)$?
  - Suppose $X_1, \ldots, X_n$ are IID outputs from CDF F.
  - Estimate F by the **empirical CDF (ECDF)** $\widehat{F}_n$:

$$\begin{aligned} \widehat{F}_n(x) &= (\text{fraction of outputs} \leq x) \\ &= \frac{1}{n} \sum_{j=1}^{n} I(X_j \leq x) \end{aligned}$$

    - ▷ $\widehat{F}_n$ is piecewise constant function with jumps of height $1/n$ at each output X_j
    - ▷ $\widehat{F}_n \approx F$ for large n by Glivenko-Cantelli Theorem: $\max_{x \in \mathbf{R}} |\widehat{F}_n(x) - F(x)| \to 0$ as $n \to \infty$ w.p.1
  - An estimator of ξ is the **sample p-quantile**:

$$\widehat{\xi}_n \;=\; \widehat{F}_n^{-1}(p) \;=\; \min\{x : \widehat{F}_n(x) \geq p\}$$





Quantile Estimation

- Equivalent way of defining sample p-quantile $\hat{\xi}_n = \hat{F}_n^{-1}(p)$
  - Sort IID $X_1, X_2, \ldots, X_n$ as **order statistics**
$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$$
  - Then sample p-quantile is
$$\hat{\xi}_n = X_{(\lceil np \rceil)}$$
so $\hat{\xi}_n$ is the $\lceil np \rceil$-th smallest output in the sample $X_1, \ldots, X_n$, where $\lceil x \rceil$ denotes round-up function.
- $\hat{\xi}_n$ satisfies a **LLN**: $\hat{\xi}_n \approx \xi$ for large sample sizes n since $\hat{F}_n \approx F$ so $\hat{\xi}_n = \hat{F}_n^{-1}(p) \approx F^{-1}(p) = \xi$.
- How can we construct confidence interval for ξ ?
  - **Complication:** $\hat{\xi}_n$ is not a sample average (nor a function of sample averages)

Quantile Estimation: CLT

- Recall: $\xi = F^{-1}(p)$ is the p-quantile of CDF F
- Assume F has density $f = F'$ and $f(\xi) > 0$.
- Since $p = F(\xi)$ and $\hat{\xi}_n \approx \xi$, a Taylor approximation gives
$$\begin{aligned} p &\approx F(\hat{\xi}_n) \\ &\approx F(\xi) + f(\xi)\left(\hat{\xi}_n - \xi\right) \\ &\approx \hat{F}_n(\xi) + f(\xi)\left(\hat{\xi}_n - \xi\right) \end{aligned}$$
since $F(\xi) \approx \hat{F}_n(\xi)$.
- Thus, rearranging terms and multiplying by $\sqrt{n}$ yields
$$\sqrt{n}\left(\hat{\xi}_n - \xi\right) \approx \frac{\sqrt{n}}{f(\xi)}\left(p - \hat{F}_n(\xi)\right) \overset{\mathcal{D}}{\approx} \mathcal{N}\left(0, \frac{p(1-p)}{f^2(\xi)}\right)$$

Quantile Estimation: CLT

- $\hat{\xi}_n$ satisfies following CLT: if $f(\xi) > 0$, then for large n,
$$\frac{\sqrt{n}}{\tau}\left(\hat{\xi}_n - \xi\right) \overset{\mathcal{D}}{\approx} \mathcal{N}(0,1)$$
with *asymptotic variance*
$$\tau^2 = \frac{p(1-p)}{f^2(\xi)}$$
- Thus, an approximate $100(1-\delta)\%$ confidence interval for ξ is
$$\left[\, \hat{\xi}_n \pm z_{1-\delta/2}\frac{\tau}{\sqrt{n}} \,\right]$$
where $z_{1-\delta/2} = \Phi^{-1}(1-\delta/2)$ and Φ is CDF of $\mathcal{N}(0,1)$.
- **Problem:** How do we estimate τ and $f(\xi)$?
  - Possible to consistently estimate τ and $f(\xi)$, but it's tricky.
  - Instead, we consider alternative approaches.

Quantile Estimation: Batching CI

- Group n IID outputs $X_1, \ldots, X_n$ into $b \ge 2$ large *batches*
  - Each batch consists of $k = n/b$ outputs

Batch 1	Batch 2	$\cdots$	Batch b
$\underbrace{X_1 \cdots X_k}_{\text{first } k}$	$\underbrace{X_{k+1} \cdots X_{2k}}_{\text{next } k}$	$\cdots$	$\underbrace{X_{(b-1)k+1} \cdots X_{bk}}_{\text{last } k}$
$\Downarrow$	$\Downarrow$		$\Downarrow$
$\tilde{\xi}_1 = \tilde{F}_1^{-1}(p)$	$\tilde{\xi}_2 = \tilde{F}_2^{-1}(p)$		$\tilde{\xi}_b = \tilde{F}_b^{-1}(p)$

- $\tilde{F}_j$ is CDF estimator from jth batch, $j = 1, 2, \ldots, b$,
$$\tilde{F}_j(x) = \frac{1}{k}\sum_{i=1}^{k} I(X_{(j-1)b+i} \le x)$$

Quantile Estimation: Batching CI

- b batch quantile estimators $\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_b$ are IID.
  - Sample average and sample variance across batches:
    $$\bar{\xi}_b = \frac{1}{b}\sum_{j=1}^{b}\tilde{\xi}_j, \qquad S_b^2 = \frac{1}{b-1}\sum_{j=1}^{b}\left(\tilde{\xi}_j - \bar{\xi}_b\right)^2.$$
  - CLT: for each batch, $\frac{\sqrt{k}}{\tau}[\tilde{\xi}_j - \xi] \overset{\mathcal{D}}{\approx} \mathcal{N}(0,1)$ for large batch size k, so each $\tilde{\xi}_j \overset{\mathcal{D}}{\approx} \mathcal{N}(\xi, \tau^2/k)$ and
    $$\frac{\bar{\xi}_b - \xi}{\sqrt{S_b^2/b}} \overset{\mathcal{D}}{\approx} T_{b-1} \quad \text{(Student } t \text{ with } b-1 \text{ d.f.)}$$
- Approximate $100(1-\delta)\%$ batching CI for ξ is
  $$\left[\bar{\xi}_b \pm \frac{t_{1-\delta/2}\, S_b}{\sqrt{b}}\right],$$
  where $t_{1-\delta/2} = (1-\delta/2)$-quantile of T_{b-1}.

Quantile Estimation: Batching CI

- Quantile estimators are **biased**:
  $$E[\hat{\xi}_n] \neq \xi$$
  for each fixed sample size n.
- Bias vanishes as $n \to \infty$, but can be significant for fixed n.
- Bias of batching estimator $\bar{\xi}_b$ is determined by batch size k:
  $$E[\bar{\xi}_b] = E\left[\frac{1}{b}\sum_{j=1}^{b}\tilde{\xi}_j\right] = \frac{1}{b}\sum_{j=1}^{b}E[\tilde{\xi}_j] = E[\tilde{\xi}_j]$$
- Since batch size $k < n/b$, batching CI is centered at very biased $\bar{\xi}_b$, so batching CI can have **poor coverage**.

Quantile Estimation: Sectioning CI

- **Sectioning** is similar to batching CI, but replaces batching estimator $\bar{\xi}_b$ with overall quantile estimator $\hat{\xi}_n = \hat{F}_n^{-1}(p)$.
- Using each batch estimator $\tilde{\xi}_j$ and overall estimator $\hat{\xi}_n$, define
  $$\tilde{S}_b^2 = \frac{1}{b-1}\sum_{j=1}^{b}\left(\tilde{\xi}_j - \hat{\xi}_n\right)^2.$$
- Approximate $100(1-\delta)\%$ sectioning CI for ξ is
  $$\left[\hat{\xi}_n \pm \frac{t_{1-\delta/2}\,\tilde{S}_b}{\sqrt{b}}\right],$$
  $t_{1-\delta/2}$ is $(1-\delta/2)$-quantile of Student t distn with $b-1$ d.f.
- Since overall sample size n is larger than batch size $k = n/b$, sectioning CI centered at less-biased $\hat{\xi}_n$ than batching CI's $\bar{\xi}_b$.
- Sectioning CI should have **better coverage** than batching CI.

Quantile Estimation: Algorithm for Sectioning CI

1. Fix $0 < p < 1$, sample size n with $np \geq 5$ and $n(1-p) \geq 5$, # batches $b = 10$, batch size $k = n/b$
2. Generate n IID outputs $X_1, X_2, \ldots, X_n \sim F$
3. Compute overall estimator $\hat{\xi}_n = \hat{F}_n^{-1}(p)$ of p-quantile ξ
   - $\hat{F}_n(x) = (1/n)\sum_{i=1}^{n} I(X_i \leq x)$ is overall CDF estimator.
4. For each batch $j = 1, 2, \ldots, b$, compute $\tilde{\xi}_j = \tilde{F}_j^{-1}(p)$
   - $\tilde{F}_j(x) = (1/k)\sum_{i=1}^{k} I(X_{(j-1)k+i} \leq x)$ is CDF estimator from outputs $X_{(j-1)b+i}$, $i = 1, 2, \ldots, k$, in batch j.
5. Approximate $100(1-\delta)\%$ sectioning CI for ξ is
   $$\left[\hat{\xi}_n \pm t_{1-\delta/2}\frac{\tilde{S}_b}{\sqrt{b}}\right]$$
   - $\tilde{S}_b^2 = (1/(b-1))\sum_{j=1}^{b}(\tilde{\xi}_j - \hat{\xi}_n)^2$
   - $t_{1-\delta/2} = (1-\delta/2)$-quantile of Student t with $b-1$ d.f.

Quantile Estimation: Remarks

- What happens when estimating p-quantile as $p \to 0$ or 1?
- Recall: p-quantile $\xi_p = F^{-1}(p)$ and estimator $\widehat{\xi}_{p,n} = \widehat{F}_n^{-1}(p)$
  - CLT: $\sqrt{n}(\widehat{\xi}_{p,n} - \xi_p) \overset{\mathcal{D}}{\approx} \mathcal{N}(0, \tau_p^2)$, with $\tau_p^2 = \frac{p(1-p)}{f^2(\xi_p)}$
- Suppose distn F has infinite support $(-\infty, \infty)$.
- Then density $f(x) \to 0$ as $x \to \pm\infty$
- As $p \to 0$ or 1, we have $\xi_p \to \pm\infty$ and $\tau_p \to \infty$.
- **Example:** If $X \sim$ Expo(1), then $\xi_p = -\ln(1-p)$ and

$$f(\xi_p) \;=\; e^{-\xi_p} \;=\; (1-p),$$

so as $p \to 1$, asymp var (AV) τ_p^2 and relative AV (RAV) blow up:

$$\tau_p^2 \;=\; \frac{p(1-p)}{f^2(\xi_p)} \;=\; \frac{p}{1-p} \;\to\; \infty \quad \text{and} \quad \text{RAV} \;=\; \frac{\tau_p^2}{\xi_p^2} \;\to\; \infty.$$

Quantile Estimation: Remarks

- When p is close to 0 or 1, then typically
  - τ_p and τ_p/ξ_p will be very large
  - Difficult to estimate ξ_p: CLT $\sqrt{n}(\widehat{\xi}_{p,n} - \xi_p) \overset{\mathcal{D}}{\approx} \mathcal{N}(0, \tau_p^2)$ implies

$$\widehat{\xi}_{p,n} \overset{\mathcal{D}}{\approx} \xi_p + \frac{\tau_p}{\sqrt{n}}\mathcal{N}(0,1)$$

  - Need very large sample size n
  - **Rule of Thumb:** choose n such that

$$np \geq 5 \quad \text{and} \quad n(1-p) \geq 5.$$

Quantile Estimation: Excel

- In Excel/PopTools, make sure to check the box "Keep results".
  - Sort results to compute $\widehat{F}_n^{-1}(p)$ as $\lceil np \rceil$th smallest value.
- In Excel/Yasai, can get outputs from all reps in **sorted order**
  - After running simulations, go to Excel menu Add-Ins > YASAI > Charts ...
  - Select Variable, and specify Chart Type as Cumulative Plot.
  - After clicking "Make Chart", sorted outputs in new worksheet "Chart Data 1"
  - For batching or sectioning, must separately simulate batches.
  - To compute overall quantile estimator for sectioning, combine outputs from all batches, and sort.

Multiple Performance Measures

Example: manufacturing system

- Terminating measures:
  - μ_1 = expected # of jobs completed in first 4 hours of day
  - μ_2 = probability ≥ 30 jobs completed in first 4 hours of day
  - μ_3 = expected average flow time during first 4 hours of day
- Use same simulation runs with n IID replications to estimate μ_1, μ_2, μ_3.
- In j-th replication, define
  - $X_{1,j}$ = number of jobs completed in first 4 hours of day
  - $X_{2,j} = I(X_{1,j} \geq 30)$
  - $X_{3,j}$ = average flow time during first 4 hours of day

Multiple Performance Measures

- For $r = 1, 2, 3$, construct approx. 95% CI I_r for μ_r
- $\mathcal{P}\{\mu_r \in I_r\} \approx 0.95$ for each $r = 1, 2, 3$.
- What is *joint* coverage of 3 CIs:

$$\mathcal{P}\{\mu_r \in I_r, \ \forall\, r = 1, 2, 3\} \ ?$$

- More generally,
  - Estimate d performance measures μ_r, $r = 1, 2, \ldots, d$
  - For each r, I_r is approx. $100(1 - \delta_r)\%$ CI for μ_r
  - What is joint coverage $\mathcal{P}\{\mu_r \in I_r, \ \forall\, r = 1, 2, \ldots, d\}$?
- **Bonferroni's inequality** :

$$\mathcal{P}\{\mu_r \in I_r, \ \forall\, r = 1, 2, \ldots, d\} \ \geq \ 1 - \sum_{r=1}^{d} \delta_r$$

Multiple Performance Measures

- If three 95% CIs I_r, $r = 1, 2, 3$, then each $\delta_r = 0.05$, so

$$\mathcal{P}\{\mu_r \in I_r, \ \forall\, r = 1, 2, 3\} \ \geq \ 1 - \sum_{r=1}^{3} \delta_r \ = \ 0.85$$

- If we want joint confidence $\geq 95\%$,
  - Might set $\delta_r = 0.01$ for each r
  - Individual 99% CIs
  - Joint probability ≥ 0.97
- If d CIs independent, then

$$\mathcal{P}\{\mu_r \in I_r, \forall\, r = 1, 2, \ldots, d\} = \prod_{r=1}^{d} \mathcal{P}\{\mu_r \in I_r\} = \prod_{r=1}^{d} (1 - \delta_r)$$

- For CIs to be indep, must use indep simulations to construct CIs (i.e., non-overlapping streams of random numbers)

Comparing 2 Systems

- Simulation is often used to compare 2 or more systems in terms of some performance measure μ.
- Suppose larger μ is better.
- Let μ_1 be the mean performance of system 1 and μ_2 be the mean performance of system 2.
- Suppose that we run simulations of both systems
  - Point estimate of μ_1 is 10
  - Point estimate of μ_2 is 14
- Which system is better?

Comparing 2 Systems

- **Answer:** We can't say because we don't know the error of each estimate.
- Now suppose
  - 95% CI for μ_1 is $[8, 12]$
  - 95% CI for μ_2 is $[11, 17]$
- Which system is better?
- **Answer:** Again, we can't say (at 95% confidence level) since CIs overlap.

Comparing 2 Systems

- Now suppose
  - 95% CI for μ_1 is $[9, 11]$
  - 95% CI for μ_2 is $[13, 15]$
- Now we can say that system 2 appears to be better.
- But how confident we that system 2 is better?
- We have roughly 95% confidence that μ_1 lies in $[9, 11]$, and 95% confidence that μ_2 lies in $[13, 15]$.
- Need to make a statement about **joint** confidence level.
  - If the two CIs are **independent**, then joint confidence is $(0.95)^2 = 0.9025$.
  - If the two CIs are **dependent**, use Bonferroni's inequality to get joint confidence is $\geq 1 - 0.05 - 0.05 = 0.90$.

Alternative Approach 1: Paired-Difference Test

- Construct CI based on **differences within pairs**.
  - Specifically, suppose we ran n IID replications of each system.
  - $X_1, X_2, \ldots, X_n$ are IID replicates from simulating system 1 with mean μ_1
  - $Y_1, Y_2, \ldots, Y_n$ are IID replicates from simulating system 2 with mean μ_2
  - X_i, Y_i **may be dependent**, but $(X_1, Y_1), \ldots, (X_n, Y_n)$ are **IID pairs**
  - **IID differences** $D_i = X_i - Y_i$, $i = 1, 2, \ldots, n$, with mean $\mu_1 - \mu_2$
  - Construct CI $= [\bar{D}_n \pm zS_D/\sqrt{n}]$ for mean difference $\mu_1 - \mu_2$ using sample mean $\bar{D}_n$ and sample variance S_D^2 of $D_1, D_2, \ldots, D_n$
- If $100(1-\delta)\%$ CI for $\mu_1 - \mu_2$ doesn't contain zero, then can conclude (at level $1 - \delta$) that one system is better than other.
  - If CI entirely above 0, then system 1 better.
  - If CI entirely below 0, then system 2 better.
  - If CI contains 0, then can't conclude one system is better than other.

Alternative Approach 2: Welch Method

- $X_1, X_2, \ldots, X_n$ are n IID outputs from system 1, mean μ_1.
- $Y_1, Y_2, \ldots, Y_m$ are m IID outputs from system 2, mean μ_2.
- Allow $m \neq n$, but X_i and Y_j must be **independent** $\forall\ i, j$.
- Compute $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$, $\bar{Y}_m = (1/m)\sum_{i=1}^m Y_i$, $S_x^2 = (1/(n-1))\sum_{i=1}^n (X_i - \bar{X}_n)^2$, $S_y^2 = (1/(m-1))\sum_{i=1}^m (Y_i - \bar{Y}_m)^2$.
- Approx. $100(1-\delta)\%$ CI for difference $\mu_1 - \mu_2$ is

$$\left[\bar{X}_n - \bar{Y}_m \ \pm\ t_{1-\delta/2,r}\left(\frac{S_x^2}{n} + \frac{S_y^2}{m}\right)^{1/2}\right],$$

where $t_{1-\delta/2,r} = (1-\delta/2)$-quantile of Student t with r d.f.,

$$r \ = \ \frac{\left[(S_x^2/n) + (S_y^2/m)\right]^2}{\frac{(S_x^2/n)^2}{n-1} + \frac{(S_y^2/m)^2}{m-1}}.$$

Optimization

- Find "best" of k systems
  - k possible designs for manufacturing system
  - Want highest expected daily production
- When k small
  - Selection procedures & multiple-comparison procedures
- When k large (possibly infinite)
  - Simulation optimization
- Ref: Henderson and Nelson, *Simulation: Handbook in OR & MS, Vol. 13*, Chapters 17–20, Elsevier, 2006.

Variance-Reduction Techniques (VRTs)

- **Common random numbers** (CRN) (Ross, Section 9.2)
  - Compare 2 systems using **same** stream of random numbers to induce **positive correlation**.
- **Antithetic variates** (Ross, Section 9.1)
  - Induce **negative correlation** by simulating replicates in pairs, e.g., using inversion with uniforms U_j and $1 - U_j$.
- **Control variates** (Ross, Section 9.2)
  - When estimating $\mu = \mathcal{E}[X]$, suppose $\exists$ rv Y with **known mean** $\nu = \mathcal{E}[Y]$, and define estimator $X + \beta(Y - \nu)$.
- **Importance sampling** (Ross, Section 9.6)
  - In rare-event simulations, **change dynamics of system** to cause events of interest to occur more frequently.
  - Unbias results by multiplying outputs by "correction factor."

Remarks on Variance-Reduction Techniques (VRTs)

- VRTs require more sophistication from user.
- CRN and antithetic variates require careful synchronization of random-number streams.
- Arena
  - 1.8×10^{19} separate streams, each of length 1.7×10^{38}.
  - EXPO(4.3, 6) uses 6th stream to generate exponential with mean 4.3
  - Stream 10 is default, so EXPO(4.3) and EXPO(4.3, 10) are same.
- Sometimes using VRTs lead to simulations taking longer
  - Make sure that reduction in variance isn't overwhelmed by extra computational effort

Derivative Estimation

- Performance measure μ is a function of the model parameters, e.g., arrival rate θ, so
$$\mu = \mu(\theta).$$
- Want to compute derivative
$$\frac{d}{d\theta}\mu(\theta)$$
  - Reliability: derivatives can help designer improve system
    - ▷ Compute partial derivatives wrt component failure rates
    - ▷ Focus on components corresponding to largest derivatives
  - Finance: hedging using "Greeks"
    - ▷ Delta, gamma, vega are sensitivities of option price wrt initial price or volatility
  - Optimization

Derivative Estimation

- Recall
$$\frac{d}{d\theta}\mu(\theta) = \lim_{h\to 0}\frac{\mu(\theta+h)-\mu(\theta)}{h} = \lim_{h\to 0}\frac{\mu(\theta+h)-\mu(\theta-h)}{2h}$$
- Methods for estimating derivatives
  - Finite differences
$$\frac{\mu(\theta+h_0)-\mu(\theta)}{h_0} \quad \text{or} \quad \frac{\mu(\theta+h_0)-\mu(\theta-h_0)}{2h_0}$$
    for small fixed $h_0 > 0$, which lead to biased estimators.
  - Perturbation analysis
  - Likelihood-ratio or score-function method
  - Ref: Chapter 19 of Henderson and Nelson (2006).

Summary

- Reviewed methods for statistical analysis of simulation output
  - Construct point estimate
  - Construct confidence interval
- Most methods are only *asymptotically* valid.
  - Need large sample sizes.
- Many other methods available.
  - Asmussen and Glynn, *Stochastic Simulation: Algorithms and Analysis*, Springer, 2007.
  - Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer-Verlag, 2004.
  - Henderson and Nelson, *Simulation: Handbook in OR & MS, Vol. 13*, Elsevier, 2006.