i under the environmental state **T**, then after simulating the value of the random vector **T** one should then evaluate, for that value of **T**, all the returns $g_i(T)$.

Appendix: Verification of Antithetic Variable Approach When Estimating the Expected Value of Monotone Functions

The following theorem is the key to showing that the use of antithetic variables will lead to a reduction in variance in comparison with generating a new independent set of random numbers whenever the function h is monotone in each of its coordinates.

Theorem *If $X_1, \ldots, X_n$ are independent, then for any increasing functions f and g of n variables*

$$E[f(\mathbf{X})g(\mathbf{X})] \geq E[f(\mathbf{X})]E[g(\mathbf{X})] \tag{8.16}$$

where $\mathbf{X} = (X_1, \ldots, X_n)$.

Proof The proof is by induction on n. To prove it when $n = 1$, let f and g be increasing functions of a single variable. Then for any x and y

$$[f(x) - f(y)][g(x) - g(y)] \geq 0$$

since if $x \geq y$ ($x \leq y$) then both factors are nonnegative (nonpositive). Hence, for any random variables X and Y,

$$[f(X) - f(Y)][g(X) - g(Y)] \geq 0$$

implying that

$$E\{[f(X) - f(Y)][g(X) - g(Y)]\} \geq 0$$

or, equivalently,

$$E[f(X)g(X)] + E[f(Y)g(Y)] \geq E[f(X)g(Y)] + E[f(Y)g(X)]$$

If we now suppose that X and Y are independent and identically distributed then, as in this case,

$$E[f(X)g(X)] = E[f(Y)g(Y)]$$
$$E[f(X)g(Y)] = E[f(Y)g(X)] = E[f(X)]E[g(X)]$$

we obtain the result when $n = 1$.

So assume that Equation (8.16) holds for $n - 1$ variables, and now suppose that $X_1, \ldots, X_n$ are independent and f and g are increasing functions. Then

$$\begin{aligned}
E[f(\mathbf{X})g(\mathbf{X})|X_n = x_n] &= E[f(X_1, \ldots, X_{n-1}, x_n)g(X_1, \ldots, X_{n-1}, x_n)|X_n = x] \\
&= E[f(X_1, \ldots, X_{n-1}, x_n)g(X_1, \ldots, X_{n-1}, x_n)] \\
&\quad \text{by independence} \\
&\geq E[f(X_1, \ldots, X_{n-1}, x_n)]E[g(X_1, \ldots, X_{n-1}, x_n)] \\
&\quad \text{by the induction hypothesis} \\
&= E[f(\mathbf{X})|X_n = x_n]E[g(\mathbf{X})|X_n = x_n]
\end{aligned}$$

Hence,

$$E[f(\mathbf{X})g(\mathbf{X})|X_n] \geq E[f(\mathbf{X})|X_n]E[g(\mathbf{X})|X_n]$$

and, upon taking expectations of both sides,

$$\begin{aligned}
E[f(\mathbf{X})g(\mathbf{X})] &\geq E[E[f(\mathbf{X})|X_n]E[g(\mathbf{X})|X_n]] \\
&\geq E[f(\mathbf{X})]E[g(\mathbf{X})]
\end{aligned}$$

The last inequality follows because $E[f(\mathbf{X})|X_n]$ and $E[g(\mathbf{X})|X_n]$ are both increasing functions of X_n, and so, by the result for $n = 1$,

$$\begin{aligned}
E[E[f(\mathbf{X})|X_n]E[g(\mathbf{X})|X_n]] &\geq E[E[f(\mathbf{X})|X_n]]E[E[g(\mathbf{X})|X_n]] \\
&= E[f(\mathbf{X})]E[g(\mathbf{X})]
\end{aligned}$$

■

Corollary *If $h(x_1, \ldots, x_n)$ is a monotone function of each of its arguments, then, for a set $U_1, \ldots, U_n$ of independent random numbers,*

$$\text{Cov}[h(U_1, \ldots, U_n), h(1 - U_1, \ldots, 1 - U_n)] \leq 0$$

Proof By redefining h we can assume, without loss of generality, that h is increasing in its first r arguments and decreasing in its final $n - r$. Hence, letting

$$f(x_1, \ldots, x_n) = h(x_1, \ldots, x_r, 1 - x_{r+1}, \ldots, 1 - x_n)$$
$$g(x_1, \ldots, x_n) = -h(1 - x_1, \ldots, 1 - x_r, x_{r+1}, \ldots, x_n)$$

it follows that f and g are both increasing functions. Thus, by the preceding theorem,

$$\text{Cov}[f(U_1, \ldots, U_n), g(U_1, \ldots, U_n)] \geq 0$$

or, equivalently,

$$\text{Cov}[h(U_1, \ldots, U_r, 1 - U_{r+1}, \ldots, 1 - U_n), h(1 - U_1, \ldots, 1 - U_r, U_{r+1}, \ldots, U_n)] \leq 0$$

The result now follows since the random vector $h(U_1, \ldots, U_n)$, $h(1 - U_1, \ldots, 1 - U_n)$ has the same joint distribution as does the random vector

$$h(U_1, \ldots, U_r, 1 - U_{r+1}, \ldots, 1 - U_n), h(1 - U_1, \ldots, 1 - U_r, U_{r+1}, \ldots, U_n) \quad \blacksquare$$

Exercises

1. Suppose we wanted to estimate θ, where

$$\theta = \int_0^1 e^{x^2}\, dx$$

Show that generating a random number U and then using the estimator $e^{U^2}(1 + e^{1-2U})/2$ is better than generating two random numbers U_1 and U_2 and using $[\exp(U_1^2) + \exp(U_2^2)]/2$.

2. Explain how antithetic variables can be used in obtaining a simulation estimate of the quantity

$$\theta = \int_0^1 \int_0^1 e^{(x+y)^2}\, dy\, dx$$

Is it clear in this case that using antithetic variables is more efficient than generating a new pair of random numbers?

3. Let X_i, $i = 1, \ldots, 5$, be independent exponential random variables each with mean 1, and consider the quantity θ defined by

$$\theta = P\left\{\sum_{i=1}^{5} iX_i \geq 21.6\right\}$$

(a) Explain how we can use simulation to estimate θ.
(b) Give the antithetic variable estimator.
(c) Is the use of antithetic variables efficient in this case?

4. Show that if X and Y have the same distribution then $\text{Var}[(X + Y)/2] \leq \text{Var}(X)$, and conclude that the use of antithetic variables can never increase variance (although it need not be as efficient as generating an independent set of random numbers).

5. **(a)** If Z is a unit normal random variable, design a study using antithetic variables to estimate $\theta = E[Z^3 e^Z]$.
(b) Using the above, do the simulation to obtain an interval of length no greater than 0.1 that you can assert, with 95 percent confidence, contains the value of θ.

6. Suppose that X is an exponential random variable with mean 1. Give another random variable that is negatively correlated with X and that is also exponential with mean 1.

7. Verify Equation (8.1).

8. Verify Equation (8.2).

9. Explain why it is unusual to obtain a large increase in efficiency from using both control variables and antithetic variables as opposed to just using one or the other.

10. Let U_n, $n \geq 1$, be a sequence of independent uniform (0, 1) random variables. Define

$$S = \min(n: U_1 + \cdots + U_n > 1)$$

It can be shown that S has the same distribution as does N in Example 3e, and so $E[S] = e$. In addition, if we let

$$T = \min(n: 1 - U_1 + \cdots + 1 - U_n > 1)$$

then it can be shown that $S + T$ has the same distribution as does $N + M$ in Example 3e. This suggests the use of $(S + T + N + M)/4$ to estimate e. Use simulation to estimate $\text{Var}(N + M + S + T)/4$.

11. In certain situations a random variable X, whose mean is known, is simulated so as to obtain an estimate of $P\{X \leq a\}$ for a given constant a. The raw simulation estimator from a single run is I, where

$$I = \begin{cases} 1 & \text{if } X \leq a \\ 0 & \text{if } X > a \end{cases}$$

Because I and X are clearly negatively correlated, a natural attempt to reduce the variance is to use X as a control—and so use an estimator of the form $I + c(X - E[X])$.

(a) Determine the percentage of variance reduction over the raw estimator I that is possible (by using the best c) if X were uniform on (0, 1).
(b) Repeat (a) if X were exponential with mean 1.
(c) Explain why we knew that I and X were negatively correlated.

12. Show that $\text{Var}(\alpha X + (1 - \alpha)W)$ is minimized by α being equal to the value given in Equation (8.3) and determine the resulting variance.