Monte Carlo Estimation of Economic Capital

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Credit portfolio
- \( m = 10^3 \) or \( 10^4 \) obligors: loans, bonds, etc., subject to default
- Obligors dependent
- Determine capital to protect against large losses with high probability.

**Goal:** use Monte Carlo to estimate economic capital \( \eta = \xi - \mu \)
- \( \xi = F^{-1}(p) \) is \( p \)-quantile or value-at-risk (VaR) of loss CDF \( F \).
- Deutsche Bank (2018): \( p = 0.999 \) or 0.9998
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Loss $Y = c(X) \sim F$ over some time horizon (e.g., 1 year)

- $c : \mathbb{R}^d \rightarrow \mathbb{R}$, with $\mathbb{R}^d$-valued $X \sim G$.
- Factor model: Glasserman & Li (2005), Bassamboo et al. (2008)

Unknown

- **CDF:** $F$ with derivative $f$ (when it exists)
- **Mean:** $\mu = \mathbb{E}[Y]$
- **$p$-quantile (value-at-risk):** $\xi = F^{-1}(p) = \inf\{ x : F(x) \geq p \}$
- **Economic capital (EC):** $\eta = \xi - \mu$

Klaassen & van Eeghen (2009), Lütkebohmert (2009), Scandizzo (2016)

AKA credit, relative or mean-adjusted VaR: Jorion (2003, 2007), McNeil et al. (2015)
Simple Random Sampling (SRS)

- Generate inputs $X_1, X_2, \ldots, X_n$ i.i.d. from $G$, compute loss $Y_i = c(X_i) \sim F$.

<table>
<thead>
<tr>
<th>Estimand</th>
<th>Expression</th>
<th>SRS Estimator</th>
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<tbody>
<tr>
<td>Mean</td>
<td>$\mu = \mathbb{E}[Y]$</td>
<td>$\hat{\mu}<em>{SRS,n} = \frac{1}{n} \sum</em>{i=1}^{n} Y_i$</td>
</tr>
<tr>
<td>CDF</td>
<td>$F(y) = P(Y \leq y) = \mathbb{E}[I(Y \leq y)]$</td>
<td>$\hat{F}<em>{SRS,n}(y) = \frac{1}{n} \sum</em>{i=1}^{n} I(Y_i \leq y)$</td>
</tr>
<tr>
<td>$p$-quantile</td>
<td>$\xi = F^{-1}(p)$</td>
<td>$\hat{\xi}<em>{SRS,n} = \hat{F}</em>{SRS,n}^{-1}(p)$</td>
</tr>
<tr>
<td>EC</td>
<td>$\eta = \xi - \mu$</td>
<td>$\hat{\eta}<em>{SRS,n} = \hat{\xi}</em>{SRS,n} - \hat{\mu}_{SRS,n}$</td>
</tr>
</tbody>
</table>

- $\hat{\eta}_{SRS,n}$ satisfies CLT as $n \to \infty$. 

Li, Kaplan, & Nakayama (NJIT)
Importance Sampling (IS)

- \( \hat{\eta}_{SRS,n} = \hat{\xi}_{SRS,n} - \hat{\mu}_{SRS,n} \) has large variance, \( p \approx 1 \).
- Recall: \( Y = c(X) \sim F, \ X \sim G \).
- **Importance Sampling (IS)** [Glynn 1996]
  - Sample \( X \sim H \) so event of interest more likely.
  - Unbias results by multiplying by correction factor.

- Rewrite **tail CDF** \( 1 - F(y) = \mathbb{E}[I(Y > y)] \) using change of measure

\[
1 - F(y) = \mathbb{E}_G \left[ I \left( c(X) > y \right) \right] = \int I \left( c(x) > y \right) dG(x) \\
= \int I \left( c(x) > y \right) \frac{dG(x)}{dH(x)} dH(x) = \mathbb{E}_H \left[ I \left( c(X) > y \right) L(X) \right]
\]

where \( L(x) = \frac{dG(x)}{dH(x)} \) is likelihood ratio (LR).
Importance Sampling (IS)

- IS algorithm: generate $X_1, X_2, \ldots, X_n$ i.i.d. $H$

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<th>IS Estimator</th>
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<tr>
<td>CDF</td>
<td>$F(y) = 1 - \mathbb{E}_H [I(c(X) &gt; y) L(X)]$</td>
<td>$\hat{F}<em>{IS,n}(y) = 1 - \frac{1}{n} \sum</em>{i=1}^{n} I(c(X_i) &gt; y) L(X_i)$</td>
</tr>
<tr>
<td>$p$-quantile</td>
<td>$\xi = F^{-1}(p)$</td>
<td>$\hat{\xi}<em>{IS,n} = \hat{F}</em>{IS,n}^{-1}(p)$</td>
</tr>
<tr>
<td>Mean</td>
<td>$\mu = \mathbb{E}_G [c(X)] = \mathbb{E}_H [c(X) L(X)]$</td>
<td>$\hat{\mu}<em>{IS,n} = \frac{1}{n} \sum</em>{i=1}^{n} c(X_i) L(X_i)$</td>
</tr>
<tr>
<td>EC</td>
<td>$\eta = \xi - \mu$</td>
<td>$\hat{\eta}<em>{IS,n} = \hat{\xi}</em>{IS,n} - \hat{\mu}_{IS,n}$</td>
</tr>
</tbody>
</table>

- $\hat{\eta}_{IS,n}$ obeys CLT as $n \to \infty$. 

Li, Kaplan, & Nakayama (NJIT)
Methods that Combine IS and SRS

- SRS: Estimates $\mu$ well, but $\xi$ poorly
- IS: Estimates $\xi$ well, but $\mu$ poorly
- Combine IS and SRS
  - Measure-specific IS (MSIS) [Shahabuddin et al. 1988]
  - IS with defensive mixture (ISDM) [Hesterberg 1995, Owen & Zhou 2000]
  - Double estimator (DE)
Measure-Specific Importance Sampling (MSIS)

- **Measure-specific IS (MSIS)** [Shahabuddin et al. 1988]
  - Estimate $\xi$ using IS.
  - Independently estimate $\mu$ using SRS.
- Fix overall sample size $n$ and allocation $\delta \in (0, 1)$.

<table>
<thead>
<tr>
<th>Method</th>
<th>Sample Size</th>
<th>Estimators</th>
</tr>
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<tbody>
<tr>
<td>IS</td>
<td>$\delta n$</td>
<td>$\hat{\xi}_{\text{IS},\delta n}$</td>
</tr>
<tr>
<td>SRS</td>
<td>$(1 - \delta)n$</td>
<td>$\hat{\mu}_{\text{SRS},(1-\delta)n}$</td>
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</tbody>
</table>

- MSIS EC estimator $\hat{\eta}_{\text{MSIS},n} = \hat{\xi}_{\text{IS},\delta n} - \hat{\mu}_{\text{SRS},(1-\delta)n}$
  - CLT as $n \to \infty$. 
Importance Sampling with a Defensive Mixture (ISDM)

- Problem with IS: \( L(x) = \frac{dG(x)}{dH(x)} \) can be huge.
- Instead sample \( X \) from mixture distribution:
  \[ X \sim H_{\text{ISDM}} = \delta H + (1 - \delta) G \]
  
  [Hesterberg 1995, Owen and Zhou 2000]

- IS with defensive mixture (ISDM)

  \[ L_{\text{ISDM}}(x) = \frac{dG(x)}{dH_{\text{ISDM}}(x)} = \frac{dG(x)}{\delta dH(x) + (1 - \delta) dG(x)} \leq \frac{1}{1 - \delta} \]

- ISDM algorithm: generate \( X_1, X_2, \ldots, X_n \) i.i.d. \( H_{\text{ISDM}} \)
  
  - Estimate both \( \xi \) and \( \mu \) from ISDM data.

- ISDM EC estimator \( \hat{\eta}_{\text{ISDM},n} = \hat{\xi}_{\text{ISDM},n} - \hat{\mu}_{\text{ISDM},n} \)
  
  - CLT: special case of IS
Double Estimator (DE)

* Use **both** IS and SRS to estimate **both** $\xi$ and $\mu$.

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<tr>
<td>IS</td>
<td>$\delta n$</td>
<td>$\hat{\xi}<em>{IS,\delta n}$ and $\hat{\mu}</em>{IS,\delta n}$</td>
</tr>
<tr>
<td>SRS</td>
<td>$(1 - \delta)n$</td>
<td>$\hat{\xi}<em>{SRS,(1-\delta)n}$ and $\hat{\mu}</em>{SRS,(1-\delta)n}$</td>
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**DE: linear combination** of the 4 estimators using weights $\nu_1, \nu_2 \in [0, 1]$,

$$
\hat{\eta}_{DE,n} = \left[ \nu_1 \hat{\xi}_{IS,\delta n} + (1 - \nu_1) \hat{\xi}_{SRS,(1-\delta)n} \right] - \left[ \nu_2 \hat{\mu}_{IS,\delta n} + (1 - \nu_2) \hat{\mu}_{SRS,(1-\delta)n} \right]
$$

- **CLT** as $n \to \infty$.
- Derived optimal weights $\nu_1, \nu_2$ to minimize $\text{Var}[\hat{\eta}_{DE,n}]$. 

Li, Kaplan, & Nakayama (NJIT)
Asymptotic Analysis of i.i.d. Sum Model

- Compare 5 methods
  - **SRS** Simple random sampling
  - **IS** Importance sampling
  - **MSIS** Measure-specific importance sampling
  - **ISDM** IS with defensive mixture
  - **DE** Double estimator

- **Loss:** \( Y \equiv Y_m = \sum_{k=1}^{m} X_k \sim F_m \) with density \( f_m \)
  - \( X_k \sim G_0 \) light tailed
  - \( Q_0(\theta) = \ln \mathbb{E}[e^{\theta X_k}] \) is CGF of \( G_0 \)
  - \( Q_0'(\theta) = \frac{d}{d\theta} Q_0(\theta) \)

- **EC** \( \eta_m = \xi_m - \mu_m \)
- Analyze as \( m \to \infty \)
  - Quantile level \( p \equiv p_m = 1 - e^{-\beta m} \), fixed \( \beta > 0 \) [Glynn 1996]

- **IS via exponential twist**
  - i.i.d. \( X_k \sim \tilde{G}_{0,\theta} \), \( d\tilde{G}_{0,\theta}(x) = e^{\theta x - Q_0(\theta)} dG_0(x) \)
  - Glynn (1996): Estimate \( \xi_m \) with \( \theta = \theta^\ast \) as root of
    \[-\theta^\ast Q_0'(\theta^\ast) + Q_0(\theta^\ast) = -\beta\]
Asymptotic Analysis of i.i.d. Sum Model

- MSIS, ISDM, DE: fixed $\delta, \nu_1, \nu_2 \in (0, 1)$ as $m \to \infty$.
- For generic estimand $\varphi_m$, compare estimators $\hat{\varphi}_m$ in terms of relative error (RE)

$$\text{RE}[\hat{\varphi}_m] = \frac{\sqrt{\text{Var}[\hat{\varphi}_m]}}{|\varphi_m|}$$

- Approximate RE ($\tilde{\text{RE}}$) for EC and $\xi$
  - Quantile approximation [Glynn 1996]
    \[\xi_m = mQ_0'(\theta_x), \quad \text{which satisfies} \quad \frac{\xi_m - \xi_m}{m} \to 0 \quad \text{as} \quad m \to \infty\]
  - Saddlepoint approximation [Jensen 1995] to density $f_m$
    \[\tilde{f}_m(x) = \frac{1}{\sqrt{2\pi mQ''_0(\theta_x)}} \exp\left[mQ_0(\theta_x) - x\theta_x\right], \quad \text{for} \quad mQ'_0(\theta_x) = x\]
    - $f_m(\xi_m)$ appears in $\text{Var}[\hat{\xi}_m]$ and $\text{Var}[\hat{\eta}_m]$
Suppose loss \( Y_m = \sum_{k=1}^{m} X_k \), and quantile level \( p_m = 1 - e^{-\beta m} \), \( \beta > 0 \).

Under regularity conditions, the estimators satisfy the following as \( m \to \infty \):

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean ( \mu_m (\theta \neq 0) )</th>
<th>Quantile ( \xi_m (\theta = \theta_\star) )</th>
<th>EC ( \eta_m (\theta = \theta_\star) )</th>
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<tbody>
<tr>
<td>SRS</td>
<td>( O(m^{-1/2}) )</td>
<td>( \text{Expo} \uparrow )</td>
<td>( \text{Expo} \uparrow )</td>
</tr>
<tr>
<td>IS</td>
<td>( \text{Expo} \uparrow )</td>
<td>( O(m^{-1/2}) )</td>
<td>( \text{Expo} \uparrow )</td>
</tr>
<tr>
<td>MSIS</td>
<td>( O(m^{-1/2}) )</td>
<td>( O(m^{-1/2}) )</td>
<td>( O(m^{-1/2}) )</td>
</tr>
<tr>
<td>ISDM</td>
<td>( O(1) )</td>
<td>( O(m^{-1/2}) )</td>
<td>( O(1) )</td>
</tr>
<tr>
<td>DE</td>
<td>( \text{Expo} \uparrow )</td>
<td>( \text{Expo} \uparrow )</td>
<td>( \text{Expo} \uparrow )</td>
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</table>

“\( \text{Expo} \uparrow \)” = exponentially increasing in \( m \)
Numerical (Non-Simulation) Results: i.i.d. sum model with $X_k \sim \text{exponential}(1)$

- SRS
- IS$(\theta_*)$
- MSIS$(\theta_*)$
- ISDM$(\theta_*)$
- DE$(\theta_*)$

$\text{RE}[\hat{\eta}_m]$ or $\text{RE}[\hat{\eta}_m]$

$10^0$  $10^1$

$10^0$  $10^1$

- SRS: Approx
- IS$(\theta_*)$: Approx
- MSIS$(\theta_*)$: Approx
- ISDM$(\theta_*)$: Approx
Credit portfolio with \( m = 1000 \) dependent obligors, 10 factors, Gaussian copula

- EC with quantile level \( p = 0.999 \)
- IS: modification of Glasserman and Li (2005) for estimating \( P(Y > x) \)
- Root-mean-square relative error (RMSRE) \( \sqrt{\mathbb{E}[(\hat{\eta}_n - \eta)^2]/\eta} \) for \( n = 2000 \)
- Coverage of nominal 95% confidence intervals (10^3 indep reps)
  - IS results unreliable: coverage \( \approx 0.05 \)
Feng & Staum (2017), Dong, Feng & Nelson (2018), …

Goal: for \( c : \mathbb{R}^d \to \mathbb{R} \) "expensive" to compute, estimate

\[
\mu(\theta) \equiv \mathbb{E}_{G_\theta} [c(X)] = \int c(x) \, dG_\theta(x), \quad \forall \theta \in \Theta.
\]

Idea: **reuse** existing data \((c(X_i), X_i)\) for \(X_i \sim G_{\theta_0}\) by change of measure

\[
\mu(\theta) = \int c(x) \frac{dG_\theta(x)}{dG_{\theta_0}(x)} \, dG_{\theta_0}(x) = \int c(x) \frac{dG_\theta(x)}{dG_{\text{ISDM},\theta,\theta_0}(x)} \, dG_{\text{ISDM},\theta,\theta_0}(x)
\]

to get unbiased estimator of \(\mu(\theta)\), where

\[
G_{\text{ISDM},\theta,\theta_0} = \delta G_{\theta_0} + (1 - \delta) G_\theta
\]

i.i.d. sum \(c(X) = \sum_{k=1}^m X_k\) for any \(\theta \neq \theta_0\) as \(m \to \infty\):

<table>
<thead>
<tr>
<th>Method</th>
<th>RE of (\hat{\mu}_m(\theta))</th>
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<tbody>
<tr>
<td>IS</td>
<td>Expo ↑</td>
</tr>
<tr>
<td>ISDM</td>
<td>(O(1))</td>
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</table>
Concluding Remarks

- EC: \( \eta = F^{-1}(p) - \mu \) for \( p \approx 1 \)
- i.i.d. sum model: theoretical analysis
  - MSIS has **vanishing** relative error (RE), and ISDM has **bounded** RE
  - SRS, IS, and DE have **unbounded** RE
- Portfolio credit risk model with dependent obligors
  - Similar empirical behavior
- “Green simulation”

Questions?