# Permuted Product and Importance-Sampling Estimators for Regenerative Simulations

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February 26, 1999

## Abstract

In a previous paper we introduced a new variance-reduction technique for regenerative simulations based on permuting regeneration cycles. In this paper we apply this idea to new classes of estimators. In particular, we derive permuted versions of likelihood-ratio derivative estimators for steady-state performance measures, importance-sampling estimators of the mean cumulative reward until hitting a set of states, and Tin estimators for steady-state ratio formulas. Empirical results are presented that show significant variance reductions in some cases.

## 1 Introduction

In Calvin and Nakayama (1998a) we introduced a new class of estimators based on permutations for regenerative simulations. The estimators were proven to have no greater (and typically strictly smaller) variance than the standard estimator. Also, the biases of the new and standard estimators are the same. The basic idea of the method is to run a simulation of a fixed number of regenerative cycles corresponding to one sequence of regeneration points, permute the cycles corresponding to a second sequence of regeneration points to obtain a new sample path, and compute an estimate based on the new path. Our new estimator is the average of the estimates computed for all possible permutations. When implementing the method, one does not have to compute all of the permutations, as we derived closed-form expressions for the estimators in some particular cases. Specifically, we gave the formulae for permuted estimators of the second moment of the cumulative reward over a regenerative cycle, the time-average variance constant, and the mean cumulative reward until hitting a set of states (when no importance sampling is used). To implement our method, the additional computational effort is typically very small and the extra storage requirements are constant (just a few extra accumulators are needed). In Calvin and Nakayama (1997) we studied the asymptotic properties of our permuted estimators for the second moment of the cumulative cycle reward and for the expected cumulative reward until some set of states is hit. In particular, we proved that these estimators satisfy strong laws of large numbers and central limit theorems.

We now derive new permuted estimators for classes of other performance measures for regenerative simulations. Specifically, we construct permuted versions of likelihood-ratio derivative estimators of steady-state performance measures, importance-sampling estimators of the mean cumulative reward until hitting a set of states, and Tin (1965) estimators (for reducing bias of ratio estimators). (The time-average variance constant also fits into our current framework.) We also discuss how to construct asymptotically valid confidence intervals based on some of these estimators.

We ran some experiments with our new permuted estimators. When estimating derivatives using the likelihood-ratio method, we found that permuting can lead to significant variance reductions. These results are of particular interest in light of the fact that (standard) likelihood-ratio derivative estimators are often plagued with large variances; e.g., see Glynn (1990). Also, we ran experiments estimating the time-average variance constant of a queueing network with 8 stations.

The rest of the paper is organized as follows. We review the basic idea of permuted regenerative estimators in Section 2. In Section 3, we derive the permuted estimator for the expectation of a product of two quantities defined over a regenerative cycle. We use this representation to derive the permuted likelihood-ratio derivative estimators and the permuted Tin estimator in Sections 4 and 6, respectively. Section 5 contains our work on permuted importance-sampling estimators. Empirical results are given in Section 7. We give some conclusions in Section 8. One of the longer proofs is deferred to Section 9. Selected results from this paper are presented (without proofs) in Calvin and Nakayama (1998b).

# 2 Basic Idea

We now describe the basic idea of our approach; for more details, see Calvin and Nakayama (1998a). Let $X = \{X(t) : t \geq 0\}$ be a continuous-time stochastic process having sample paths that are right continuous with left limits on a state space $S \subset \Re^d$. We can handle discrete-time processes $\{X_n : n = 0, 1, 2, \ldots\}$ in this framework by letting $X(t) = X_{\lfloor t \rfloor}$ for all $t \geq 0$, where $\lfloor a \rfloor$ is the greatest integer less than or equal to $a$.

Let $T = \{T(i) : i = 0, 1, 2, \ldots\}$ be an increasing sequence of nonnegative finite stopping times. Consider the random pair $(X, T)$ and the shift

$$\theta_{T(i)}(X, T) = ((X(T(i) + t)_{t \geq 0}, \, (T(k) - T(i))_{k \geq i}) \, .$$

We define the pair $(X, T)$ to be a *regenerative process* (in the classic sense) if

**(i)** $\{\theta_{T(i)}(X, T) : i = 0, 1, 2, \ldots\}$ are identically distributed;

**(ii)** for each $i \geq 0$, $\theta_{T(i)}(X, T)$ does not depend on the "prehistory"

$$\Big( (X(t))_{t < T(i)}, \, T(0), T(1), \ldots, T(i) \Big) \, .$$

See p. 19 of Kalashnikov (1994) for more details. This definition allows for so-called delayed regenerative processes (e.g., Section 2.6 of Kingman 1972).

We assume that there are two distinct increasing sequences of nonnegative finite stopping times, $T_1 = \{T_1(i) : i = 0, 1, 2, \ldots\}$ with $T_1(0) = 0$ and $T_2 = \{T_2(i) : i = 0, 1, 2, \ldots\}$, such that $(X, T_1)$ and $(X, T_2)$ are both regenerative processes. For example, if $X$ is an irreducible, positive-recurrent, discrete-time or continuous-time Markov chain on a countable state space $S$, then we can define $T_1$ and $T_2$ to be the sequences of hitting times to the states $v \in S$ and $w \in S$, respectively, where we assume that $X(0) = v$ and $w \neq v$. We use the terminology that for $l = 1, 2$, the $k$th $T_l$-cycle is the segment of the sample path from time $T_l(k-1)$ to $T_l(k)$.

The simplest way of explaining the technique developed in Calvin and Nakayama (1998a) is as follows. Suppose that we want to estimate some performance measure $\alpha$, and we have available a "standard" estimator of $\alpha$, which we will call $\widehat{\alpha}_m(\vec{X})$, based on a sample path $\vec{X}$ of a fixed number $m$ of $T_1$-cycles. To construct our permuted estimator, we first construct a new sample path $\vec{X}'$ from $\vec{X}$ by permuting the $T_2$-cycles in the path. More specifically, let $M_2$ be the number of occurrences of stopping times from the sequence $T_2$ that occur during the $m$ $T_1$-cycles in the path $\vec{X}$. Note that if $M_2 = 0$ or 1, then the path $\vec{X}$ has no $T_2$-cycles. If $M_2 = 2$, then there is only one $T_2$-cycle. Assume now that $M_2 \geq 3$. Then for the given path $\vec{X}$, we can now look at the $(M_2 - 1)$ $T_2$-cycles in the path. We generate a uniform random permutation of the $(M_2 - 1)$ $T_2$-cycles within the path $\vec{X}$, and this gives us our new sample path $\vec{X}'$, which also has $m$ $T_1$-cycles. Calvin and Nakayama (1998a) show that $\vec{X}' \stackrel{\mathcal{D}}{=} \vec{X}$, where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution.

Now for the new sample path $\vec{X}'$, we can calculate $\widehat{\alpha}_m(\vec{X}')$, which is just the estimate obtained from the new sample path $\vec{X}'$. The number of possible paths $\vec{X}'$ we can construct from $\vec{X}$ by permuting $T_2$-cycles is $N(\vec{X}) = (M_2 - 1)!$, which depends on $\vec{X}$ and is therefore random. We label these paths $\vec{X}^{(1)}$ $(= \vec{X}), \vec{X}^{(2)}, \ldots, \vec{X}^{(N(\vec{X}))}$, each of which has the same distribution as $\vec{X}$, and for each one we compute $\widehat{\alpha}_m(\vec{X}^{(i)})$. We finally define our new estimator for $\alpha$ to be

$$\widetilde{\alpha}_m(\vec{X}) = \frac{1}{N(\vec{X})} \sum_{i=1}^{N(\vec{X})} \widehat{\alpha}_m(\vec{X}^{(i)}). \tag{1}$$

Below, we derive closed-form expressions for some specific permuted estimators $\widetilde{\alpha}_m(\vec{X})$, and so the user does not have to compute all permutations to apply our method.

A different way of looking at the permuted estimator $\widetilde{\alpha}_m(\vec{X})$ in (1) is as follows. Given the original path $\vec{X}$, let $P_*$ be a probability measure on $\Omega_* = \{\vec{X}^{(1)}, \vec{X}^{(2)}, \ldots, \vec{X}^{(N(\vec{X}))}\}$ that assigns probability $1/N(\vec{X})$ to each permuted path, and let $E_*$ be the corresponding expectation. Then $\widetilde{\alpha}_m(\vec{X})$ is the expectation of $\widehat{\alpha}_m(\vec{X}')$ given $\vec{X}$, where $\vec{X}'$ is chosen from $\Omega_*$ using the measure $P_*$; i.e.,

$$\widetilde{\alpha}_m(\vec{X}) = E\left[\widehat{\alpha}_m(\vec{X}') \middle| \vec{X}\right] = E_*\left[\widehat{\alpha}_m(\vec{X}')\right]. \tag{2}$$

The new estimator has the same mean as the original since

$$E[\widetilde{\alpha}_m(\vec{X})] = E\left[\, E_*[\widehat{\alpha}_m(\vec{X}')]\,\right] = E[\widehat{\alpha}_m(\vec{X}')] = E[\widehat{\alpha}_m(\vec{X})]$$

because $\vec{X}' \stackrel{\mathcal{D}}{=} \vec{X}$. Moreover, decomposing the variance by conditioning on $\vec{X}$ gives us

$$\mathrm{Var}\left(\widehat{\alpha}_m(\vec{X})\right) = \mathrm{Var}\left(\widehat{\alpha}_m(\vec{X}')\right) = \mathrm{Var}\left(E_*[\widehat{\alpha}_m(\vec{X}')]\right) + E\left[\,\mathrm{Var}(\,\widehat{\alpha}_m(\vec{X}') \mid \vec{X})\,\right].$$

Thus, $E\left[\,\mathrm{Var}(\,\widehat{\alpha}_m(\vec{X}') \mid \vec{X})\,\right] \geq 0$ implies that the variance of the new estimator $\widetilde{\alpha}_m(\vec{X}) = E_*\left[\widehat{\alpha}_m(\vec{X}')\right]$ is no greater than that of the original estimator $\widehat{\alpha}_m(\vec{X})$, and so the mean squared error of our new estimator $\widetilde{\alpha}_m(\vec{X})$ is no greater than that of the original estimator $\widehat{\alpha}_m(\vec{X})$. Strict variance reduction is obtained unless $\mathrm{Var}(\,\widehat{\alpha}_m(\vec{X}') \mid \vec{X}) = 0$ with probability one. These results were previously shown in Calvin and Nakayama (1998a).

# 3 Permuted Product Estimators

We first consider estimating

$$\alpha = E[U(1)V(1)], \tag{3}$$

where $U(k)$ and $V(k)$, $k = 1, 2, \ldots$, are defined over the $k$th $T_1$-cycle and have a structure to be described below. The standard estimator of $\alpha$ is

$$\widehat{\alpha}_m(\vec{X}) = \frac{1}{m} \sum_{k=1}^{m} U(k)V(k), \tag{4}$$

and we want to derive a closed-form formula for the permuted estimator of $\alpha$. The reason we consider performance measures of this type is that likelihood-ratio derivative estimators, Tin estimators and estimators of the time-average variance constant include quantities having the form in (4).

We now describe the assumed form of $U(k)$ and $V(k)$. For our two sequences of stopping times $T_1$ and $T_2$, let $H(1;2) \subset \{1, 2, \ldots, m\}$ denote the set of indices of the $T_1$-cycles in which a $T_2$ stopping time occurs, and define the complementary set $J(1;2) = \{1, 2, \ldots, m\} - H(1;2)$. More specifically, $H(1;2) = \{i \leq m : T_1(i-1) \leq T_2(j) < T_1(i) \text{ for some } j\}$. We analogously define the set $H(2;1)$ with the roles of $T_1$ and $T_2$ reversed. For $k \in H(1;2)$, define $T_2'(k) = \inf\{t > T_1(k-1) : T_2(l) = t \text{ for some } l\}$, which is the first occurrence of a stopping time from sequence $T_2$ after the $(k-1)$st stopping time from the sequence $T_1$. Similarly define $\widetilde{T}_2(k) = \sup\{t < T_1(k) : T_2(l) = t \text{ for some } l\}$, which is the last occurrence of a stopping time from sequence $T_2$ before the $k$th occurrence of the stopping-time sequence $T_1$. We define $B_k \subset J(2;1)$ to be the set of indices of those $T_2$-cycles that do not contain any occurrences of the stopping times from the sequence $T_1$ and that are between $T_1(k-1)$ and $T_1(k)$. Then we assume that for each $T_1$-cycle index $k \in H(1;2)$, $U(k)$ and $V(k)$ can be expressed as

$$U(k) = U_{12}(k) + \sum_{l \in B_k} U_{22}(l) + U_{21}(k), \tag{5}$$

$$V(k) = V_{12}(k) + \sum_{l \in B_k} V_{22}(l) + V_{21}(k), \tag{6}$$

where for $k \in H(1;2)$, $U_{12}(k)$ is the contribution to $U(k)$ from the beginning of the $k$th $T_1$-cycle until a stopping time from sequence $T_2$ occurs, $U_{21}(k)$ is the contribution to $U(k)$ from the last occurrence of a stopping time from sequence $T_2$ in the $k$th $T_1$-cycle until the end of the cycle, and for $l \in J(2;1)$, $U_{22}(l)$ is the contribution from the $l$th $T_2$-cycle in which there is no occurrence of a stopping time from sequence $T_1$; the variables $V_{12}(k)$, $V_{21}(k)$, and $V_{22}(l)$ are defined similarly relative to $V(k)$. For example, this situation arises when $U(k)$ and $V(k)$ are additive functionals of the sample path over the $k$th $T_1$-cycle, such as

$$U(k) = \int_{T_1(k-1)}^{T_1(k)} f(X(t))dt, \tag{7}$$

where $f : S \to \Re$ is some "reward" function, in which case,

$$U_{12}(k) = \int_{T_1(k-1)}^{T_2'(k)} f(X(t))dt, \tag{8}$$

$$U_{21}(k) = \int_{\widetilde{T}_2(k)}^{T_1(k)} f(X(t))dt, \tag{9}$$

$$U_{22}(l) = \int_{T_2(l-1)}^{T_2(l)} f(X(t))dt. \tag{10}$$

To write out our permuted estimator of $\alpha$, we need more notation. Let $h_{12} = |H(1;2)|$. Define $\beta_l$ to be the $l$th smallest element of the set $H(1;2)$ for $l = 1, 2, \ldots, h_{12}$, and define $\beta_0 = \beta_{h_{12}}$. For $k = \beta_l \in H(1;2)$ for some $l = 1, 2, \ldots, h_{12}$, define $\psi(k) = \beta_{l-1}$; i.e., $\psi(k)$ is the index in $H(1;2)$ that occurs just before $k$ if $k$ is not the first index, and $\psi(k)$ is the last element in $H(1;2)$ if $k$ is the first element. Then the following holds, whose proof appears in Section 9.

**Theorem 1** *Suppose we want to estimate $\alpha$ defined in (3) with $U(k)$ and $V(k)$, $k \in H(1;2)$, satisfying (5) and (6), respectively, and assume that $E[U(1)^2 V(1)^2] < \infty$. Then, our new estimator is given by*

$$
\begin{aligned}
\tilde{\alpha}_m(\vec{X}) &= \frac{1}{m}\Bigg[\sum_{k \in J(1;2)} U(k)V(k) + \sum_{k \in H(1;2)} \left(U_{12}(k)V_{12}(k) + U_{21}(k)V_{21}(k)\right) \\
&+ \frac{1}{h_{12}-1} \sum_{k \in H(1;2)} U_{12}(k) \left( \sum_{j \in H(1;2)} V_{21}(j) - V_{21}(\psi(k)) \right) \\
&+ \frac{1}{h_{12}-1} \sum_{k \in H(1;2)} V_{12}(k) \left( \sum_{j \in H(1;2)} U_{21}(j) - U_{21}(\psi(k)) \right) \\
&+ \frac{1}{h_{12}} \sum_{k \in H(1;2)} \left(U_{12}(k) + U_{21}(k)\right) \sum_{j \in J(2;1)} V_{22}(j) \\
&+ \frac{1}{h_{12}} \sum_{k \in H(1;2)} \left(V_{12}(k) + V_{21}(k)\right) \sum_{j \in J(2;1)} U_{22}(j) \\
&+ \sum_{l \in J(2;1)} U_{22}(l)V_{22}(l) + \frac{2}{1+h_{12}} \sum_{\substack{i,j \in J(2;1) \\ i \neq j}} U_{22}(i)V_{22}(j)\Bigg]
\end{aligned} \tag{11}
$$

*if $M_2 \geq 3$, and $\tilde{\alpha}_m(\vec{X}) = \hat{\alpha}_m(\vec{X})$ otherwise, where $\hat{\alpha}_m(\vec{X})$ is the standard estimator of $\alpha$ as defined in (4). The estimator satisfies $E[\tilde{\alpha}_m(\vec{X})] = \alpha$ and $Var(\tilde{\alpha}_m(\vec{X})) \leq Var(\hat{\alpha}_m(\vec{X}))$.*

## 3.1 Computational Issues

To compute the permuted estimator in (11), we do not have to store the entire sample path. Instead, we only need to keep track of a few extra accumulators. To see why this is the case, note that

$$
\begin{aligned}
&\sum_{k \in H(1;2)} U_{12}(k) \left( \sum_{j \in H(1;2)} V_{21}(j) - V_{21}(\psi(k)) \right) \\
&= \sum_{k \in H(1;2)} U_{12}(k) \sum_{j \in H(1;2)} V_{21}(j) - \sum_{k \in H(1;2)} U_{12}(k) V_{21}(\psi(k)).
\end{aligned}
$$

Also,

$$\sum_{\substack{i,j\in J(2;1)\\ i\neq j}} U_{22}(i)V_{22}(j) = \left(\sum_{i\in J(2;1)} U_{22}(i)\right)\left(\sum_{j\in J(2;1)} V_{22}(j)\right) - \sum_{l\in J(2;1)} U_{22}(l)V_{22}(l).$$

Thus, we need to have accumulators for

- $U(k)V(k)$ for $k \in J(1;2)$;
- $U_{12}(k), U_{21}(k), V_{12}(k), V_{21}(k), U_{12}(k)V_{12}(k), U_{21}(k)V_{21}(k), U_{12}(k)V_{21}(\psi(k)), V_{12}(k)U_{21}(\psi(k))$ for $k \in H(1;2)$;
- $U_{22}(l)$, $V_{22}(l)$, and $U_{22}(l)V_{22}(l)$ for $l \in J(2;1)$.

All of these terms can then be combined to compute $\widetilde{\alpha}_m(\vec{X})$ in (11). Note that the amount of storage required is fixed and does not increase as the simulation proceeds.

# 4 Permuted Derivative Estimators

Consider the steady-state performance measure

$$\alpha = \lim_{t\to\infty} \frac{1}{t}\int_0^t g(X(s))ds = \frac{E[Y(g;1)]}{E[\tau(1)]} \tag{12}$$

for some function $g : S \to \Re$, where for $T_1$-cycle indices $k = 1, 2, 3, \ldots$,

$$\begin{aligned} Y(g;k) &= \int_{T_1(k-1)}^{T_1(k)} g(X(s))ds, \\ \tau(k) &= T_1(k) - T_1(k-1). \end{aligned}$$

Suppose our goal now is to estimate the derivative of $\alpha$ with respect to some system parameter $\lambda$, and assume that $g = g_\lambda$ is continuously differentiable in $\lambda$. For example, $\alpha$ may be the steady-state availability of a reliability system, and we are interested in computing its derivative with respect to the failure rate $\lambda$ of some component. Assume that the random times $T_1(0), T_1(1), \ldots$ do not depend on $\lambda$ (but their distribution may depend on $\lambda$). Differentiating the ratio formula in (12) yields

$$\partial\alpha = \frac{(\partial E[Y(g;1)])E[\tau(1)] - E[Y(g;1)](\partial E[\tau(1)])}{E^2[\tau(1)]}, \tag{13}$$

where we use the notation $\partial A$ to denote the derivative of $A$ with respect to $\lambda$. Under appropriate regularity conditions, applying the likelihood-ratio method of differentiation gives

$$\partial E[Y(g;1)] = E[Y(g;1)\partial L(1)] + E[\partial Y(g;1)]$$

and

$$\partial E[\tau(1)] = E[\tau(1)\partial L(1)],$$

where

$$\partial Y(g;k) = \int_{T_1(k-1)}^{T_1(k)} \partial g(X(t))\,dt$$

for $k \geq 1$, and $\partial L(1)$ is the derivative of the likelihood ratio; see Glynn (1990) for more details. For example, when $X$ is a discrete-time Markov chain on a countable state space $S$ with transition matrix $R = \{R(x,y) : x, y \in S\}$, the derivative of the likelihood ratio over the $k$th $T_1$-cycle is

$$\partial L(k) = \sum_{j=T_1(k-1)}^{T_1(k)-1} \frac{\partial R(X_j, X_{j+1})}{R(X_j, X_{j+1})}.$$

The standard likelihood-ratio-method estimator of $\partial\alpha$ is

$$\widehat{\partial\alpha}_m(\vec{X}) = \frac{1}{\left(\sum_{k=1}^m \tau(k)\right)^2} \left( \sum_{i=1}^m (Y(g;i)\partial L(i) + \partial Y(g;i)) \sum_{j=1}^m \tau(j) - \sum_{k=1}^m Y(g;k) \sum_{l=1}^m \tau(l)\partial L(l) \right). \tag{14}$$

## 4.1 Permuted Point Estimator

To compute our permuted derivative estimator, first note that $\sum_{k=1}^m \tau(k)$, $\sum_{k=1}^m Y(g;k)$, and $\sum_{k=1}^m \partial Y(g;k)$ are unchanged by permutations of $T_2$-cycles since they are linear functionals of the sample path. Thus, the permuted derivative estimator is given by

$$\begin{aligned}
\widetilde{\partial\alpha}_m(\vec{X}) &= E_*\left[\widehat{\partial\alpha}_m(\vec{X}')\right] \\
&= \frac{1}{\left(\sum_{k=1}^m \tau(k)\right)^2} \left( E_*\left[\sum_{i=1}^m Y'(g;i)\partial L'(i)\right] \sum_{j=1}^m \tau(j) \right. \\
&+ \left. \sum_{i=1}^m \partial Y(g;i) \sum_{j=1}^m \tau(j) - \sum_{k=1}^m Y(g;k) E_*\left[\sum_{l=1}^m \tau'(l)\partial L'(l)\right] \right),
\end{aligned} \tag{15}$$

where $Y'(g;i)$, $\partial L'(i)$ and $\tau'(l)$ are the same as $Y(g;i)$, $\partial L(i)$ and $\tau(l)$, respectively, except the primed variables are defined with respect to the permuted path $\vec{X}'$. In general, we can express

$$\partial L(k) = \partial L_{12}(k) + \sum_{l \in B_k} \partial L_{22}(l) + \partial L_{21}(k);$$

see Glynn (1990). Also, $Y(g;k)$ has precisely the form of $U(k)$ in (7) with the function $f = g$, and so for $k \in H(1;2)$, we can express

$$Y(g;k) = Y_{12}(g;k) + \sum_{l \in B_k} Y_{22}(g;l) + Y_{21}(g;k), \tag{16}$$

where $Y_{12}(g;k)$, $Y_{21}(g;k)$, and $Y_{22}(g;l)$ are the same as $U_{12}(k)$, $U_{21}(k)$, and $U_{22}(l)$ as defined in (8)–(10), respectively, with $f = g$. Also, by defining in (7) the function $f \equiv 1$, we get that for $k \in H(1;2)$,

$$\tau(k) = \tau_{12}(k) + \sum_{l \in B_k} \tau_{22}(l) + \tau_{21}(k), \tag{17}$$

where $\tau_{12}(k)$, $\tau_{21}(k)$, and $\tau_{22}(l)$ are the same as $U_{12}(k)$, $U_{21}(k)$, and $U_{22}(l)$ as defined in (8)–(10), respectively, with $f \equiv 1$. Hence, using (11), we can compute $E_* \left[\sum_{k=1}^m Y'(g;k) \partial L'(k)\right]$ with $U(k) = Y(g;k)$ and $V(k) = \partial L(k)$, and we can compute $E_* \left[\sum_{k=1}^m \tau'(k) \partial L'(k)\right]$ with $U(k) = \tau(k)$ and $V(k) = \partial L(k)$. The next result then follows from Theorem 1.

**Corollary 2** *Suppose we want to estimate $\partial\alpha$ defined in (13), and assume that $E[\widehat{\partial}\alpha_m(\vec{X})^2] < \infty$, where $\widehat{\partial}\alpha_m(\vec{X})$ is the standard (non-permuted) likelihood-ratio derivative estimator defined in (14). Then, our new estimator $\widetilde{\partial}\alpha_m(\vec{X})$ is given by (15) if $M_2 \geq 3$, and $\widetilde{\partial}\alpha_m(\vec{X}) = \widehat{\partial}\alpha_m(\vec{X})$ otherwise. The estimator satisfies $E[\widetilde{\partial}\alpha_m(\vec{X})] = E[\widehat{\partial}\alpha_m(\vec{X})]$ and $Var(\widetilde{\partial}\alpha_m(\vec{X})) \leq Var(\widehat{\partial}\alpha_m(\vec{X}))$.*

Thus, we see that our permuted derivative estimator has no greater mean-squared error (MSE) than the standard derivative estimator. In our empirical results in Section 7, we see that the MSE is often significantly lower for the permuted estimator.

## 4.2 Confidence Intervals

We now state without proof a central limit theorem for our permuted derivative estimators. (The result can be established by applying the techniques used in Calvin and Nakayama 1997.) This will allow us to construct confidence intervals for $\partial\alpha$.

To state the central limit theorem, we need some additional notation. For $T_1$-cycle indices $k = 1, 2, \ldots$, define

$$\begin{aligned}
Z_1(k) &= 1\{k \in H(1;2)\}, \\
Z_2(k) &= Y(g;k), \\
Z_3(k) &= Y(g;k)\, \partial L(k)\, 1\{k \in J(1;2)\}, \\
Z_4(k) &= \tau(k), \\
Z_5(k) &= \tau(k)\, \partial L(k)\, 1\{k \in J(1;2)\}, \\
Z_6(k) &= \partial Y(g;k), \\
Z_7(k) &= Y_{12}(g;k)\, 1\{k \in H(1;2)\}, \\
Z_8(k) &= Y_{21}(g;k)\, 1\{k \in H(1;2)\}, \\
Z_9(k) &= \tau_{12}(k)\, 1\{k \in H(1;2)\},
\end{aligned}$$

$$
\begin{aligned}
Z_{10}(k) &= \tau_{21}(k)\, 1\{k \in H(1;2)\}, \\
Z_{11}(k) &= \partial L_{12}(k)\, 1\{k \in H(1;2)\}, \\
Z_{12}(k) &= \partial L_{21}(k)\, 1\{k \in H(1;2)\}, \\
Z_{13}(k) &= Y_{12}(g;k)\, \partial L_{12}(k)\, 1\{k \in H(1;2)\}, \\
Z_{14}(k) &= Y_{21}(g;k)\, \partial L_{21}(k)\, 1\{k \in H(1;2)\}, \\
Z_{15}(k) &= \tau_{12}(k)\, \partial L_{12}(k)\, 1\{k \in H(1;2)\}, \\
Z_{16}(k) &= \tau_{21}(k)\, \partial L_{21}(k)\, 1\{k \in H(1;2)\}, \\
Z_{17}(k) &= \sum_{l \in B_k} Y_{22}(g;l)\, 1\{k \in H(1;2)\}, \\
Z_{18}(k) &= \sum_{l \in B_k} \tau_{22}(l)\, 1\{k \in H(1;2)\}, \\
Z_{19}(k) &= \sum_{l \in B_k} \partial L_{22}(l)\, 1\{k \in H(1;2)\}, \\
Z_{20}(k) &= \sum_{l \in B_k} Y_{22}(g;l)\, \partial L_{22}(l)\, 1\{k \in H(1;2)\}, \\
Z_{21}(k) &= \sum_{l \in B_k} \tau_{22}(l)\, \partial L_{22}(l)\, 1\{k \in H(1;2)\},
\end{aligned}
$$

where $1\{A\}$ denotes the indicator function of the event $A$; i.e., $1\{A\} = 1$ if $A$ occurs, and 0 otherwise. Also, define $Z(k) = (Z_1(k), Z_2(k), \ldots, Z_{21}(k))^T$, $k = 1, 2, \ldots$, where the superscript $T$ denotes transpose, and note that $Z(k)$ is a random vector defined over the $k$th $T_1$-cycle. Hence, $Z(1), Z(2), \ldots$ are i.i.d. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_{21})^T = E[Z(1)]$, and let $\Sigma = (\sigma_{i,j} : i, j = 1, 2, \ldots, 21)$ be the covariance matrix of $Z(1)$, where $\sigma_{i,j} = \mathrm{Cov}(Z_i(1), Z_j(1))$. For $i = 1, 2, \ldots, 21$, define the sample means

$$
\bar{Z}_i(m) = \frac{1}{m} \sum_{k=1}^{m} Z_i(k)
$$

and define $\bar{Z}(m) = (\bar{Z}_1(m), \bar{Z}_2(m), \ldots, \bar{Z}_{21}(m))^T$.

Now define the function $f : \Re^{21} \to \Re$ to be

$$
\begin{aligned}
& f(z_1, z_2, \ldots, z_{21}) \\
&= \frac{1}{z_4^2} \Bigg[ \left( z_3 + z_{13} + z_{14} + \frac{z_7 z_{12}}{z_1} + \frac{z_{11} z_8}{z_1} + \frac{(z_7 + z_8) z_{19}}{z_1} + \frac{(z_{11} + z_{12}) z_{17}}{z_1} + z_{20} + \frac{2 z_{17} z_{19}}{z_1} + z_6 \right) z_4 \\
&- \left( z_5 + z_{15} + z_{16} + \frac{z_9 z_{12}}{z_1} + \frac{z_{11} z_{10}}{z_1} + \frac{(z_9 + z_{10}) z_{19}}{z_1} + \frac{(z_{11} + z_{12}) z_{18}}{z_1} + z_{21} + \frac{2 z_{18} z_{19}}{z_1} \right) z_2 \Bigg].
\end{aligned}
$$

Define the functions $f_i(z_1, z_2, \ldots, z_{21}) = \partial f(z_1, z_2, \ldots, z_{21}) / \partial z_i$ for $i = 1, 2, \ldots, 21$, and define the column vector $\nabla f(z) = (f_1(z), f_2(z), \ldots, f_{21}(z))^T$, which is the gradient of $f$ evaluated at $z$. Then define $\sigma_{\partial\alpha}^2 = \nabla f(\mu)^T \, \Sigma \, \nabla f(\mu)$. Now applying arguments similar to those used in

Calvin and Nakayama (1997), we have that under the appropriate moment conditions,

$$\widetilde{\partial}\alpha_m \to \partial\alpha \quad \text{a.s.} \tag{18}$$

and

$$\sqrt{m}(\widetilde{\partial}\alpha_m - \partial\alpha) \Rightarrow N(0, \sigma^2_{\partial\alpha}) \tag{19}$$

as $m \to \infty$, where $\Rightarrow$ denotes convergence in distribution and $N(x, y)$ denotes a normal random variable with mean $x$ and variance $y$.

The basic idea of the proofs of (18) and (19) is as follows. We can express

$$\widetilde{\partial}\alpha_m = f(\bar{Z}(m)) + R(m),$$

where $R(m)$ is some random variable. Then we can show that $R(m) \to 0$ a.s. and that $f(\bar{Z}(m)) \to \partial\alpha$ a.s., from which (18) follows. Also, using the corollary on p. 124 of Serfling (1980), we get that $\sqrt{m}(f(\bar{Z}(m)) - \partial\alpha) \Rightarrow N(0, \sigma^2_{\partial\alpha})$, and we can also show that $\sqrt{m}R(m) \Rightarrow 0$. Applying Theorem 4.4.6 in Chung (1974) then establishes (19).

To construct confidence intervals for $\partial\alpha$ using the estimator $\widetilde{\partial}\alpha_m$, we estimate $\nabla f(\mu)$ and the covariance $\sigma_{i,j}$ by $\nabla f(\bar{Z}(m))$ and

$$\hat{\sigma}_{i,j}(m) = \frac{1}{m-1}\sum_{k=1}^{m}\left(Z_i(k) - \bar{Z}_i(m)\right)\left(Z_j(k) - \bar{Z}_j(m)\right),$$

respectively, and let $\hat{\Sigma}(m) = (\hat{\sigma}_{i,j}(m) : i, j = 1, 2, \ldots, 21)$. Then an approximate $100(1-\delta)\%$ confidence interval for $\partial\alpha$ is

$$\left[\widetilde{\partial}\alpha_m - \frac{\kappa_\delta \hat{\sigma}_{\partial\alpha}(m)}{\sqrt{m}},\ \widetilde{\partial}\alpha_m + \frac{\kappa_\delta \hat{\sigma}_{\partial\alpha}(m)}{\sqrt{m}}\right],$$

where $\hat{\sigma}^2_{\partial\alpha}(m) = \nabla f(\bar{Z}(m))^T\,\hat{\Sigma}(m)\,\nabla f(\bar{Z}(m))$ and $\kappa_\delta$ is the upper $\delta/2$ critical point of a standard normal distribution; i.e., $P\{N(0,1) \le \kappa_\delta\} = 1 - \delta/2$.

# 5 Permuted Importance-Sampling Estimators

Suppose we now are interested in computing

$$\eta = E\left[\int_0^{T_F} g(X(t))\,dt\right], \tag{20}$$

where $T_F = \inf\{t > 0 : X(t) \in F\}$ for some set of states $F \subset S$, and $g : S \to \Re$ is some function. Thus, $\eta$ is the expected cumulative reward until hitting $F$ conditional on $T_1(0) = 0$.

By letting $g \equiv 1$ and letting $F$ be a set of "failed" states in a reliability system, we see that the mean time to failure is a special case of $\eta$. It can be shown that

$$\eta = \frac{\xi}{\gamma}, \tag{21}$$

where

$$\xi = E\left[\int_0^{T_F \wedge T_1(1)} g(X(t))\, dt\right]$$

and

$$\gamma = E\left[1\{T_F < T_1(1)\}\right], \tag{22}$$

with $a \wedge b = \min(a, b)$; e.g., see Goyal et al. (1992).

To estimate $\eta$, we generate sample paths $\vec{X}_1$ and $\vec{X}_2$, each consisting of $m$ $T_1$-cycles, and we use $\vec{X}_1$ to estimate $\xi$ and $\vec{X}_2$ to estimate $\gamma$. We can either let $\vec{X}_1 = \vec{X}_2$ or take $\vec{X}_1$ independent of $\vec{X}_2$. In the context of rare-event simulation, one may estimate the numerator $\xi$ and the denominator $\gamma$ independently, where importance sampling is applied in the estimation of $\gamma$ and standard simulation is used to estimate $\xi$. This is known as measure-specific importance sampling (Goyal et al. 1990), and it is the approach we will take here here.

## 5.1 Permuted Point Estimator

In Calvin and Nakayama (1998a), we derive the permuted estimator of $\xi$ when using no importance sampling (which we give below), so now we apply our permutation method to an importance-sampling estimator of $\gamma$. To do this, let $P$ be the (original) probability measure associated with the process $X$, and let $P_0$ be another probability measure that is absolutely continuous with respect to $P$; see p. 442 of Billingsley (1986). Let $E_0$ denote the expectation operator associated with $P_0$. The basic idea of importance sampling is to use the following change of measure:

$$\gamma = E[1\{T_F < T_1(1)\}] = E_0[1\{T_F < T_1(1)\}L],$$

where $L(\omega) = dP(\omega)/dP_0(\omega)$ is the likelihood ratio for a sample path $\omega$ over a single $T_1$-cycle. Then we estimate $\gamma$ by using the measure $P_0$ to generate i.i.d. observations $(I(1), L(1))$, $(I(2), L(2)), \ldots$ of $(1\{T_F < T_1(1)\}, L)$. The (non-permuted) estimator for $\gamma$ based on importance sampling is then

$$\widehat{\gamma}_0(\vec{X}_2) = \frac{1}{m}\sum_{k=1}^{m} I(k)L(k) = \frac{1}{m}\left(\sum_{k \in J(1;2)} I(k)L(k) + \sum_{k \in H(1;2)} I(k)L(k)\right). \tag{23}$$

For example, consider a discrete-time Markov chain $X = \{X_n : n = 0, 1, 2, \ldots\}$ on a countable state space $S$ with transition matrix $R = \{R(x, y) : x, y \in S\}$. We can define our importance-sampling measure $P_0$ to be Markovian with transition matrix $R_0 = \{R_0(x,y) : x,y \in S\}$, with the requirement that $R_0(x,y) = 0$ implies that $R(x,y) = 0$. Then the likelihood ratio over the $k$th $T_1$-cycle is

$$L(k) = \prod_{j=T_1(k-1)}^{T_1(k)-1} \frac{R(X_j, X_{j+1})}{R_0(X_j, X_{j+1})}.$$

See Glynn and Iglehart (1989) for more details on importance sampling.

The first term on the far-right-hand side in (23) is independent of permutations of $T_2$-cycles, but the second is not, and so we now focus on the latter term. Since $L$ is multiplicative, we can break up $L(k)$ for each $k \in H(1;2)$ by dividing the sample path into segments as before. Let $L_{12}(k)$ be the likelihood ratio from the beginning of the $k$th $T_1$-cycle in $H(1;2)$ until a stopping time from sequence $T_2$ occurs. Let $L_{21}(k)$ be the likelihood ratio from the last occurrence of a stopping time from sequence $T_2$ in the $k$th cycle in $H(1;2)$ until the end of the cycle. Also, for $l \in J(2;1)$, let $L_{22}(l)$ be the likelihood ratio over the $l$th $T_2$-cycle in which there is no occurrence of a stopping time from sequence $T_1$. Then we have

$$L(k) = L_{12}(k) L_{21}(k) \prod_{l \in B_k} L_{22}(l).$$

We now want to similarly express $I(k)$ as a function of variables over the different segments in the $k$th $T_1$-cycle. For $k \in H(1;2)$, let

$$\begin{aligned} I_{12}(k) &= 1\{T'_F(k) < T'_2(k)\}, \\ I_{22}(l) &= 1\{T_F^{(2)}(l) < T_2(l)\}, \\ I_{21}(k) &= 1\{T_F^{(1)}(k) < T_1(k)\}, \end{aligned}$$

with $T'_F(k) = \inf\{t > T_1(k-1) : X(t) \in F\}$, $T'_2(k) = \inf\{t > T_1(k-1) : T_2(i) = t \text{ for some } i\}$, $T_F^{(2)}(l) = \inf\{t > T_2(l-1) : X(t) \in F\}$ and $T_F^{(1)}(k) = \inf\{t > \widetilde{T}_2(k) : X(t) \in F\}$. Hence, $I_{12}(k)$ (resp., $I_{21}(k)$) is the indicator of whether the set $F$ is hit in the initial 1-2 segment (resp., final 2-1 segment) of the $T_1$-cycle with index $k \in H(1;2)$. (Here, we use the terminology 1-2 segment to denote the sample path starting from an occurrence of a $T_1$-stopping time until the next $T_2$-stopping time. We similarly define 2-1 segments.) Similarly, $I_{22}(l)$ is the indicator whether the set $F$ is hit in the $T_2$-cycle with index $l \in J(2;1)$. Then note that

$$I(k) = 1 - I_{12}^c(k) I_{21}^c(k) \prod_{l \in B_k} I_{22}^c(l),$$

where the superscript $c$ denotes complement; e.g., $I_{12}^c(k) = 1 - I_{12}(k)$. Thus, for $k \in H(1;2)$,

$$I(k)L(k) = L_{12}(k) L_{21}(k) \prod_{l \in B_k} L_{22}(l) - \left(I_{12}^c(k) L_{12}(k)\right)\left(I_{21}^c(k) L_{21}(k)\right) \prod_{l \in B_k} \left(I_{22}^c(l) L_{22}(l)\right). \quad (24)$$

Using the representation of $I(k)L(k)$ in (24), we can determine how the right-hand side of (23) is altered by averaging over all permutations of $T_2$-cycles.

To write out explicitly our new estimator, we need more notation. Let $j_{21} = |J(2;1)|$, which is the number of $T_2$-cycles that do not contain an occurrence of a stopping time from sequence $T_1$. Also, for some collection of numbers $C = \{C_{22}(1), C_{22}(2), \ldots, C_{22}(j_{21})\}$, define

$$d(C) = \sum_{t=0}^{j_{21}} \frac{S_C(t)}{\binom{h_{12}+j_{21}-1}{t}(h_{12}+j_{21}-t-1)},$$

where

$$S_C(t) = \sum_{\substack{l_1,l_2,\ldots,l_t \in J(2;1):\\ l_1<l_2<\cdots<l_t}} C_{22}(l_1)C_{22}(l_2)\cdots C_{22}(l_t)$$

for $t \geq 1$, and we define $S_C(0) = 1$. Then the permuted importance-sampling estimator for $\gamma$ is

$$\begin{aligned}\tilde{\gamma}_0(\vec{X}_2) \;=\; & \frac{1}{m}\left[\sum_{k\in J(1;2)} I(k)L(k) + \sum_{k\in H(1;2)} L_{12}(k)\left(\sum_{j\in H(1;2)} L_{21}(j) - L_{21}(\psi(k))\right) d(L)\right.\\ & \left. - \sum_{k\in H(1;2)} I^c_{12}(k)L_{12}(k)\left(\sum_{j\in H(1;2)} I^c_{21}(j)L_{21}(j) - I^c_{21}(\psi(k))L_{21}(\psi(k))\right) d(I^cL)\right] \qquad (25)\end{aligned}$$

when $M_2(\vec{X}_2) \geq 3$, and $\tilde{\gamma}_0(\vec{X}_2) = \hat{\gamma}_0(\vec{X}_2)$ otherwise. It is important to note that the above estimator is defined for the sample path $\vec{X}_2$. Thus, all of the variables in the above expression are defined relative to $\vec{X}_2$.

The permuted estimator of $\xi$ (without importance sampling) based on the sample path $\vec{X}_1$ was derived in Calvin and Nakayama (1998a), and we give it here for completeness. To do this, we need to define the following quantities relative to $\vec{X}_1$. Let

$$\begin{aligned} k_{21} &= |\{k \in H(1;2) : I_{21}(k) = 0\}|, \quad k_{12} = |\{k \in H(1;2) : I_{12}(k) = 0\}|, \\ k_c &= |\{k \in H(1;2) : I_{12}(k) = 0,\, I_{21}(\psi(k)) = 0\}|. \end{aligned}$$

Define

$$\begin{aligned} D(k) &= \int_{T_1(k-1)}^{T_1(k)\wedge T'_F(k)} g(X(t))\,dt, \qquad D_{12}(k) = \int_{T_1(k-1)}^{T'_F(k)\wedge T'_2(k)} g(X(t))\,dt, \\ D_{22}(l) &= \int_{T_2(l-1)}^{T_F^{(2)}(l)\wedge T_2(l)} g(X(t))\,dt, \qquad D_{21}(k) = \int_{\widetilde{T}_2(k)}^{T_F^{(1)}(k)\wedge T_1(k)} g(X(t))\,dt. \end{aligned}$$

Finally, let

$$r = \sum_{l\in J(2;1)} I_{22}(l), \quad d_0 = \sum_{l\in J(2;1)} D_{22}(l)\,(1 - I_{22}(l))\,, \quad d_1 = \sum_{l\in J(2;1)} D_{22}(l) I_{22}(l).$$

Then Calvin and Nakayama (1997) derive the permuted estimator for $\xi$ as

$$
\begin{aligned}
\tilde{\xi}(\vec{X}_1) &= \frac{1}{m}\left(\sum_{k\in J(1;2)} D(k) + \sum_{k\in H(1;2)} D_{12}(k)\right. \\
&+ \frac{1}{h_{12}-1+r}\left(k_{12}\sum_{j\in H(1;2)} D_{21}(j) - \sum_{k\in H(1;2)} (1-I_{12}(k))D_{21}(\psi(k))\right) \\
&+ \left. k_{12}\left(\frac{d_0}{r+h_{12}} + \frac{d_1}{r+h_{12}-1}\right)\right), \qquad (26)
\end{aligned}
$$

if $M_2(\vec{X}_1) \geq 3$, and otherwise $\tilde{\xi}(\vec{X}_1) = \hat{\xi}(\vec{X}_1)$, where

$$\hat{\xi}(\vec{X}_1) = \frac{1}{m_1}\sum_{k=1}^{m_1} D(k).$$

It is important to note that the above estimator is defined for the sample path $\vec{X}_1$.

The following result gives our new estimator for $\eta$ when using measure-specific importance sampling.

**Theorem 3** *Suppose we want to estimate $\eta$ in (21) using measure-specific importance sampling with the two independent sample paths $\vec{X}_1$ and $\vec{X}_2$, where $\vec{X}_1$ (resp., $\vec{X}_2$) is generated using the original measure $P$ (resp., using importance sampling with measure $P_0$) and is used to estimate $\xi$ (resp., $\gamma$). Assume that $P_0$ is absolutely continuous with respect to $P$ and $E[I(1)L(1)^2] < \infty$. Assume that $E[D(1)^2] < \infty$. Then, the measure-specific-importance-sampling estimator of $\eta$ is given by $\tilde{\eta}_0(\vec{X}_1, \vec{X}_2) = \tilde{\xi}(\vec{X}_1)/\tilde{\gamma}_0(\vec{X}_2)$; where $\tilde{\xi}(\vec{X}_1)$ is defined by (26) if $M_2(\vec{X}_1) \geq 3$, and $\tilde{\xi}(\vec{X}_1) = \hat{\xi}(\vec{X}_1)$ otherwise; and $\tilde{\gamma}_0(\vec{X}_2)$ is defined by (25) if $M_2(\vec{X}_2) \geq 3$, and $\tilde{\gamma}_0(\vec{X}_2) = \hat{\gamma}_0(\vec{X}_2)$ otherwise. The estimator $\tilde{\gamma}_0(\vec{X}_2)$ satisfies $E[\tilde{\gamma}_0(\vec{X}_2)] = E[\hat{\gamma}_0(\vec{X}_2)]$ and $Var(\tilde{\gamma}_0(\vec{X}_2)) \leq Var(\hat{\gamma}_0(\vec{X}_2))$.*

**Proof.** We only derive (25). To do this, we have to compute the average over all permutations of $T_2$-cycles of the right-hand side of (23):

$$
\begin{aligned}
\tilde{\gamma}_0(\vec{X}_2) &= E_*\left[\frac{1}{m}\left(\sum_{k\in J(1;2)} I'(k)L'(k) + \sum_{k\in H(1;2)} I'(k)L'(k)\right)\right] \\
&= \frac{1}{m}\sum_{k\in J(1;2)} I(k)L(k) + \frac{1}{m}E_*\left[\sum_{k\in H(1;2)} I'(k)L'(k)\right],
\end{aligned}
$$

where $I'(k)$ and $L'(k)$ are the same as $I(k)$ and $L(k)$, respectively, except that the primed variables are defined with respect to the permuted path $\vec{X}_2'$. Then by (24),

$$\begin{aligned} E_*\left[\sum_{k\in H(1;2)} I'(k)L'(k)\right] &= E_*\left[\sum_{k\in H(1;2)} L_{12}(k)L_{21}(\rho(k))\prod_{l\in B_k'} L_{22}(l)\right] \\ &- E_*\left[\sum_{k\in H(1;2)} (I_{12}^c(k)L_{12}(k))\,(I_{21}^c(\rho(k))L_{21}(\rho(k)))\prod_{l\in B_k'}(I_{22}^c(l)L_{22}(l))\right], \end{aligned}$$

where $B_k'$ is the same as $B_k$ except that the primed variable is defined with respect to the permuted path $\vec{X}_2'$, and $\rho(k)$ is the index of the 2-1 segment that follows the $k$th 1-2 segment after a permutation of the $T_2$-cycles. Hence, it suffices to determine the value of

$$e(C)\equiv E_*\left[\sum_{k\in H(1;2)} C_{12}(k)C_{21}(\rho(k))\prod_{l\in B_k'} C_{22}(l)\right],$$

for some collection of $C_{12}(k)$, $C_{21}(k)$, and $C_{22}(l)$. Note that

$$\begin{aligned} e(C) &= \sum_{k\in H(1;2)} C_{12}(k)\ E_*\left[C_{21}(\rho(k))\prod_{l\in B_k'} C_{22}(l)\right] \\ &= \sum_{k\in H(1;2)} C_{12}(k)\ E_*\left[C_{21}(\rho(k))\right]\ E_*\left[\prod_{l\in B_k'} C_{22}(l)\right] \\ &= \sum_{k\in H(1;2)} C_{12}(k)\left(\frac{1}{h_{12}-1}\right)\left(\sum_{j\in H(1;2)} C_{21}(j)-C_{21}(\psi(k))\right) E_*\left[\prod_{l\in B_1} C_{22}(l)\right]. \end{aligned}$$

To compute $E_*\left[\prod_{l\in B_1} C_{22}(l)\right]$, assume we have $q$ identical black balls and $j_{21}$ distinct white balls labeled with numbers $C_{22}(1), C_{22}(2), \ldots, C_{22}(j_{21})$, where $q\equiv h_{12}-1$. Generate a random uniform permutation of the $q+j_{21}$ total balls, and the quantity $E_*\left[\prod_{l\in B_1} C_{22}(l)\right]$ can then be computed as the expectation of the product of the $C_{22}$'s starting from the left until the first black ball is encountered. The total number of permutations is $(q+j_{21})!/q!$. There are a total of $(q+j_{21}-1)!/(q-1)!$ possible ways of placing a black ball in the leftmost position, which corresponds to an empty product. Similarly, there are a total of $(q+j_{21}-2)!/(q-1)!$ ways of choosing one of the $C_{22}$'s and placing it in the leftmost position and a black ball in the second position, and this corresponds to having one term in the product. Continuing on in a similar manner yields

$$\begin{aligned} &E_*\left[\prod_{l\in B_1} C_{22}(l)\right] \\ &= \frac{(q+j_{21}-1)!/(q-1)!}{(q+j_{21})!/q!} + \sum_{t=1}^{j_{21}} \frac{(q+j_{21}-t-1)!/(q-1)!}{(q+j_{21})!/q!}\, t! \sum_{\substack{l_1,l_2,\ldots,l_t\in J(2;1):\\ l_1<l_2<\cdots<l_t}} C_{22}(l_1)C_{22}(l_2)\cdots C_{22}(l_t) \end{aligned}$$

$$= \frac{q}{(q+j_{21})!}\left[(q+j_{21}-1)! + \sum_{t=1}^{j_{21}} (q+j_{21}-t-1)!\,t! \sum_{\substack{l_1,l_2,\ldots,l_t \in J(2;1):\\ l_1<l_2<\cdots<l_t}} C_{22}(l_1)C_{22}(l_2)\cdots C_{22}(l_t)\right]$$

$$= q\sum_{t=0}^{j_{21}} \frac{S_C(t)}{\binom{q+j_{21}}{t}(q+j_{21}-t)} = (h_{12}-1)d(C).$$

Hence, replacing the $C$'s with either the $L$'s or the $I^cL$'s gives the result. ■

## 5.2 Computational Issues

We now examine some of the computational issues associated with the estimator $\tilde{\gamma}_0(\vec{X}_2)$ in (25). First note that in contrast to the permuted product estimator and permuted derivative estimator, the formula for $\tilde{\gamma}_0(\vec{X}_2)$ requires that the amount of storage increases as the simulation run length grows. Specifically, computing $d(C)$ requires $j_{21}$ quantities to be stored, and $j_{21}$ grows randomly and approximately linearly on average as the simulation proceeds. Also, the number of calculations in a naive implementation of the computation of $d(C)$ grows exponentially in $j_{21}$. However, we now show how $d(C)$ can be computed in time of order $O(j_{21}^2)$.

To do this, we will recursively update the values of $S_C(0), S_C(1), S_C(2), \ldots$. First let $S_C(0) = 1$. When the first term $C(1)$ is generated, let $S_C(1) = C(1)$. When the second term $C(2)$ is generated, let $S_C(2) = S_C(1)C(2)$, and then update $S_C(1)$ to be $S_C(1) + C(2)$. When the third term $C(3)$ is generated, let $S_C(3) = S_C(2)C(3)$, and then update $S_C(2)$ to be $S_C(2) + S_C(1)C(3)$, and then update $S_C(1)$ to be $S_C(1) + C(3)$. In general, when the $t$-th term $C(t)$ is generated, update the $S_C$'s according to the following algorithm:

$S_C(t) \leftarrow S_C(t-1)C(t)$;
for $l = 1,\ t-1$, do
    $i \leftarrow t - l$;
    $S_C(i) \leftarrow S_C(i) + S_C(i-1)C(t)$;
enddo;

## 6 Permuted Tin Estimator

Suppose our goal is to estimate some steady-state performance measure $\alpha$ having the form in (12). The standard ratio estimator is then

$$\widehat{\alpha}_m(\vec{X}) = \frac{\sum_{k=1}^m Y(g;k)}{\sum_{k=1}^m \tau(k)}.$$

Calvin and Nakayama (1998a) showed that permuting the $T_2$-cycles has no effect on the above estimator, and we now examine another estimator of $\alpha$ for which the permutation idea will result in a new estimator.

One can show that

$$E[\widehat{\alpha}_m(\vec{X})] = \alpha - \frac{1}{m}\frac{E[(Y(g;1) - \alpha\tau(1))\,\tau(1)]}{E^2[\tau(1)]} + O(m^{-2});$$

see p. 104 of Shedler (1993). Hence, the standard estimator $\widehat{\alpha}_m(\vec{X})$ is biased, and various techniques have been introduced to try to reduce the bias. One such estimator was proposed by Tin (1965), and is given by

$$\widehat{\alpha}_{m,T}(\vec{X}) = \widehat{\alpha}_m(\vec{X}) + \frac{1}{m^2}\sum_{k=1}^m \frac{\left(Y(g;k) - \widehat{\alpha}_m(\vec{X})\tau(k)\right)\tau(k)}{\bar{\tau}(m)^2}, \tag{27}$$

where $\bar{\tau}(m) = (1/m)\sum_{k=1}^m \tau(k)$.

### 6.1 Permuted Point Estimator

By averaging (27) over all permutations of $T_2$-cycles, we construct the permuted Tin estimator as

$$\begin{aligned}
\widetilde{\alpha}_{m,T}(\vec{X}) &= E_*\left[\widehat{\alpha}_{m,T}(\vec{X}')\right] \\
&= E_*\left[\widehat{\alpha}_m(\vec{X}')\right] + \frac{1}{m^2}E_*\left[\sum_{k=1}^m \frac{\left(Y'(g;k) - \widehat{\alpha}_m(\vec{X}')\tau'(k)\right)\tau'(k)}{\bar{\tau}'(m)^2}\right] \\
&= \widehat{\alpha}_m(\vec{X}) + \frac{1}{m^2\,\bar{\tau}(m)^2}\left(E_*\left[\sum_{k=1}^m Y'(g;k)\tau'(k)\right] - \widehat{\alpha}_m(\vec{X})E_*\left[\sum_{k=1}^m \tau'(k)^2\right]\right), \qquad (28)
\end{aligned}$$

where the last equation follows since both $\bar{\tau}(m)$ and $\widehat{\alpha}_m(\vec{X})$ are independent of permutations of $T_2$-cycles. Note that $Y(g;k)$ and $\tau(k)$ have the representations (16) and (17), respectively. Hence, using (11), we can compute $E_*\left[\sum_{k=1}^m Y'(g;k)\tau'(k)\right]$ with $U(k) = Y(g;k)$ and $V(k) = \tau(k)$, and we can compute $E_*\left[\sum_{k=1}^m \tau'(k)^2\right]$ with $U(k) = V(k) = \tau(k)$. Then the following is a direct consequence of Theorem 1.

**Corollary 4** *Suppose we want to estimate* $\alpha$ *defined in (12), and assume that* $E[\widehat{\alpha}_{m,T}(\vec{X})^2] < \infty$*, where* $\widehat{\alpha}_{m,T}(\vec{X})$ *is the standard Tin estimator of* $\alpha$ *as defined in (27). Then, our new estimator* $\widetilde{\alpha}_{m,T}(\vec{X})$ *is given by (28) if* $M_2 \geq 3$*, and* $\widetilde{\alpha}_{m,T}(\vec{X}) = \widehat{\alpha}_{m,T}(\vec{X})$ *otherwise. The estimator satisfies* $E[\widetilde{\alpha}_{m,T}(\vec{X})] = E[\widehat{\alpha}_{m,T}(\vec{X})]$ *and* $Var(\widetilde{\alpha}_{m,T}(\vec{X})) \leq Var(\widehat{\alpha}_{m,T}(\vec{X}))$*.*

Thus, we see that our permuted Tin estimator has the same (reduced) bias as the standard Tin estimator but with the added benefit of no greater (and often strictly smaller) variance.

### 6.2 Confidence Intervals

We now want to derive a central limit theorem for our permuted Tin estimator $\widetilde{\alpha}_{m,T}$. First, it is well known that if $E[(Y(f;1) - \alpha\tau(1))^2] < \infty$, then

$$\sqrt{m}(\widehat{\alpha}_m - \alpha) \Rightarrow N(0, \sigma_\alpha^2)$$

as $m \to \infty$, where $\sigma_\alpha^2 = E[(Y(f;1) - \alpha\tau(1))^2]/E^2[\tau(1)]$; e.g., see Theorem 1.13 of Shedler (1993). Let

$$\widetilde{\alpha}_{m,2} = \frac{1}{m\,\bar{\tau}(m)^2}\left(E_*\left[\sum_{k=1}^{m} Y(g;k)\tau(k)\right] - \widehat{\alpha}_m(\vec{X})E_*\left[\sum_{k=1}^{m} \tau(k)^2\right]\right).$$

Using methods similar to those applied in Calvin and Nakayama (1997), we can show that

$$\widetilde{\alpha}_{m,2} \to \frac{E[(Y(g;1) - \alpha\tau(1))\,\tau(1)]}{E^2[\tau(1)]}$$

with probability 1 as $m \to \infty$. (The basic idea is to express $\widetilde{\alpha}_{m,2}$ as a function of sample means of quantities defined over $T_1$-cycles and a remainder term.) Thus, $\widetilde{\alpha}_{m,2}/\sqrt{m} \Rightarrow 0$ as $m \to \infty$, and so Theorem 4.4.6 in Chung (1974) implies that $\widetilde{\alpha}_{m,T} \Rightarrow N(0, \sigma_\alpha^2)$ as $m \to \infty$. Hence, we find that asymptotically, the permuted (and non-permuted) Tin estimator has the same time-average variance constant as the standard regenerative ratio estimator. However, in the small-sample context, the (permuted or non-permuted) Tin estimator typically has smaller bias.

## 7 Empirical Results

We now present some empirical results from simulating various discrete-time Markov chains to construct some permuted estimators. In each case, we let the stopping-time sequence $T_1$ correspond to successive hitting times to some fixed state $v$, which we call the return state. The standard (non-permuted) regenerative estimator is always computed based on the sequence $T_1$. The $T_2$ sequence corresponds to hits to some state $w \neq v$, and we examine the variance reduction resulting from different choices of $w$.

### 7.1 Derivative Estimators

We first consider a discrete-time Markov chain $X = \{X_n : n = 0, 1, 2, \ldots\}$ on state space $S = \{0, 1, \ldots, M\}$ with transition probability matrix $R(i, i + 1) = \lambda/(i + \lambda) = 1 - R(i, i - 1)$ for $0 < i < M$ and $R(0, 1) = R(M, M - 1) = 1$. This chain is the discrete-time version of the Erlang loss system. As $n \to \infty$, $X_n \Rightarrow X_\infty$, where $P\{X_\infty = i\} = \pi_i$, $0 \le i \le M$, and $\pi \equiv (\pi_0, \pi_1, \ldots, \pi_M)$ is the stationary distribution of $X$, which we can compute as follows. (Note that $\pi$ depends on the parameter $\lambda$, and so we sometimes write $\pi \equiv \pi(\lambda) \equiv (\pi_0(\lambda), \pi_1(\lambda), \ldots, \pi_M(\lambda))$ to emphasize the dependence.)

Let $\rho_0 = 1$, and for $1 \le k \le M$, set

$$\rho_k = \prod_{j=0}^{k-1} \frac{R(j, j + 1)}{R(j + 1, j)} = \frac{\lambda^{k-1}}{k!}(k + \lambda).$$

Then $\pi_k = \rho_k / \sum_{j=0}^{M} \rho_j$. We consider the average cost $\alpha(\lambda) \equiv \lambda E_{\pi(\lambda)}[X_\infty]$, where $E_{\pi(\lambda)}$ is the expectation operator under the steady-state distribution $\pi$ parameterized by $\lambda$, and our goal is to estimate

$$\partial\alpha(\lambda) = E_{\pi(\lambda)}[X_\infty] + \lambda \frac{d}{d\lambda} E_{\pi(\lambda)}[X_\infty] = \sum_{i=1}^{M} i \left(\pi_i + \lambda\, \partial\pi_i\right).$$

This is equivalent to

$$\partial\alpha(\lambda) = \sum_{i=1}^{M} i\pi_i \left[ \frac{\lambda}{i + \lambda} + i - \lambda \sum_{k=1}^{M} \pi_k \left( \frac{1}{k + \lambda} + \frac{k - 1}{\lambda} \right) \right]. \tag{29}$$

We ran numerical experiments with $M = 20$ and $\lambda = 5$. The theoretical value of $\partial\alpha(5)$, calculated according to (29), is approximately 10.5. Table 1 reports the results of simulations of 1,000 independent replications for the return states $v = 0$, 5 and 10. Each column labeled with "Avg. of $\tilde{\alpha}_m$" (resp., "Samp. Var.") is the average (resp., sample variance) of the permuted estimators over the 1,000 replications. In each column the row where $w = v$ corresponds to the standard (non-permuted) estimator.

The expected $T_1$-cycle lengths for the different choices of the return state vary considerably, with $v = 5$ having the shortest expected cycle length (about 5.7) over the entire state space. When $v = 0$, the expected cycle length is quite long (about 300), and the expected cycle length for $v = 10$ is somewhere between the two (about 37). For each return state the number of $T_1$-cycles simulated is adjusted so that the total expected number of transitions in each replication is about $20,000$. For $v = 0$ as the return state, 67 cycles were simulated; for $v = 5$ as the return state, 3,501 cycles were simulated; and for $v = 10$ as the return state, 544 cycles were required. Hence, we can directly compare the results across the three tables.

| | $v = 0$ | | $v = 5$ | | $v = 10$ | |
|---|---|---|---|---|---|---|
| $w$ | Avg. of $\tilde{\alpha}_m$ | Samp. Var. | Avg. of $\tilde{\alpha}_m$ | Samp. Var. | Avg. of $\tilde{\alpha}_m$ | Samp. Var. |
| 0 | 10.35 | 2.43 | 10.47 | 0.23 | 10.44 | 0.51 |
| 1 | 10.32 | 0.88 | 10.47 | 0.22 | 10.44 | 0.36 |
| 2 | 10.30 | 0.58 | 10.48 | 0.22 | 10.44 | 0.28 |
| 3 | 10.30 | 0.45 | 10.48 | 0.22 | 10.44 | 0.23 |
| 4 | 10.29 | 0.38 | 10.47 | 0.22 | 10.45 | 0.21 |
| 5 | 10.30 | 0.36 | 10.47 | 0.23 | 10.45 | 0.19 |
| 6 | 10.30 | 0.35 | 10.48 | 0.21 | 10.44 | 0.19 |
| 7 | 10.31 | 0.38 | 10.47 | 0.20 | 10.45 | 0.21 |
| 8 | 10.31 | 0.42 | 10.47 | 0.19 | 10.43 | 0.26 |
| 9 | 10.32 | 0.51 | 10.48 | 0.19 | 10.44 | 0.38 |
| 10 | 10.31 | 0.66 | 10.48 | 0.20 | 10.43 | 0.58 |
| 11 | 10.31 | 0.96 | 10.48 | 0.21 | 10.43 | 0.59 |
| 12 | 10.34 | 1.38 | 10.48 | 0.22 | 10.43 | 0.59 |
| 13 | 10.35 | 1.90 | 10.47 | 0.22 | 10.43 | 0.59 |
| 14 | 10.32 | 2.10 | 10.47 | 0.22 | 10.43 | 0.59 |
| 15 | 10.33 | 2.33 | 10.47 | 0.22 | 10.43 | 0.58 |
| 16 | 10.35 | 2.41 | 10.47 | 0.23 | 10.43 | 0.58 |
| 17 | 10.35 | 2.43 | 10.47 | 0.23 | 10.43 | 0.58 |
| 18 | 10.35 | 2.43 | 10.47 | 0.23 | 10.43 | 0.58 |
| 19 | 10.35 | 2.43 | 10.47 | 0.23 | 10.43 | 0.58 |
| 20 | 10.35 | 2.43 | 10.47 | 0.23 | 10.43 | 0.58 |

Table 1: Derivative estimates.

Also, the difference in the number of cycles may account for the difference in the apparent bias among the columns corresponding to the different choices of return state $v$.

Previous theoretical and empirical results (e.g., see Glynn 1990, Reiman and Weiss 1989) show that the likelihood-ratio derivative method results in estimators with variances that grow approximately linearly with the expected $T_1$-cycle length. In our experiments, when $v = 0$, the expected cycle length is very long, and the sample variance of the standard derivative estimator (which corresponds to the row $w = 0$) is about 2.4. The variance can be significantly reduced by permuting the estimator, with about a factor of 7 reduction in the best case ($w = 6$). When $v = 5$ the variance reduction from permutations is modest, but the variance is significantly reduced (up to a factor of 3) when $v = 10$. Thus, we conclude that when a "bad" return state is chosen (i.e., a state corresponding to a long expected $T_1$-cycle length), permutations can significantly reduce the variance. However, if a "good" return state is used, then the improvement is more modest. This conclusion is consistent with the numerical results for other permuted estimators reported in Calvin and Nakayama (1998a). (For discussions on how to choose the two sequences $T_1$ and $T_2$, see Calvin and Nakayama 1997,1998.)

We now more closely analyze the results for $v = 0$ and $v = 5$. First, note that the ratio of the variances for the standard (non-permuted) estimators for the two choices of return state $v$ is about 11 (i.e., 2.4 to 0.23). However, when comparing (for these two return states) the variances of the best permuted estimators (i.e., $w = 6$ for $v = 0$, and $w = 9$ for $v = 5$), we see that the ratio is now about 1.8 (i.e., 0.35 to 0.19). Thus, while there seems to be a degradation of the permuted estimators when the expected cycle length is large, the amount of variance increase is significantly less than that for the non-permuted estimators. In fact, when comparing the results for $v = 5$ and $v = 10$, we find that the best permuted estimators for these two return states (i.e., $w = 5$ for $v = 10$, and $w = 9$ for $v = 5$) have about the same variance, even though the expected cycle length for $v = 10$ is more than 6 times larger than that for $v = 5$ and the variance of the standard estimator when $v = 10$ is more than twice that when $v = 5$.

We conclude this section with a discussion of the efficiency of the permuted estimators compared with the standard estimator. We define the relative efficiency of our estimator to be the reciprocal of the product of the relative variance and the relative running time, each relative to that of the standard estimator; see Glynn and Whitt (1992). The relative efficiency of the different permuted estimators depends on the choice of $w$. Numerical experiments were conducted using $v = 0$ and comparing the relative efficiency for different choices of $w$. The running times do not differ greatly; the maximum running time (for $w = 3$) was only about 4% longer than for the standard method. The maximum relative efficiency, 6.7, is attained for $w = 6$. The relative efficiency is greater than 1 for all $w \leq 15$, and the worst relative efficiency (for $w = 20$) is about 0.98. If the state $w$ is hit fewer than 3 times in the simulation (which is likely in this example for $w \geq 15$), then the permuted and standard estimators are the same; there is no decrease in variance, and a slight increase in computational overhead.

## 7.2 Estimators of the Time-Average Variance Constant

This subsection describes the results of estimating the time-average variance constant for a queueing network. The queueing model, much larger than the model of the previous section, has 8 stations labeled 1, 2, ..., 8. Customers from outside arrive to station $i$, $i = 1, 2, \ldots, 8$, according to a Poisson process with rate $\lambda_i$. Each station $i$ has a single server who serves customers in a FIFO manner with service times following an exponential distribution with rate $\mu_i$. After a customer completes service at server $i$, it is routed to server $j$ with probability $p_j$ (independent of $i$), and it leaves the system with probability

$p_0 = 1 - \sum_{j=1}^{8} p_j$. Each station has a queue (to hold waiting customers) with a capacity of 14 customers. If a customer arrives at a station and the queue is full, the customer is lost (i.e., leaves the system). We consider the embedded discrete-time Markov chain $X = \{X_n : n = 0, 1, 2, \ldots\}$, where $X_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,8})$ and $X_{n,i}$ is the number of customers (i.e., those in service and in queue) at station $i$ after the $n$th state transition (i.e., either external customer arrival or service completion). The state space of $X$ is $S = \{(n_1, n_2, \ldots, n_8) : 0 \leq n_i \leq 15,\ i = 1, 2, \ldots, 8\}$, and note that $|S| = 16^8 \approx 4 \times 10^9$. We define the function $g : S \to \Re$ as $g(n_1, n_2, \ldots, n_8) = \sum_{i=1}^{8} n_i$. The goal of our simulation experiments is to estimate the time-average variance constant $\sigma_g^2$ of $g(X)$, where $g(X) = \{g(X_n) : n = 0, 1, 2, \ldots\}$ is the discrete-time process of the total number of customers in the system after each transition.

As we shall now see, the time-average variance $\sigma_g^2$ can be estimated using the framework given in Section 3. First, define $r_n = (1/n) \sum_{j=0}^{n-1} g(X_j)$ and $r = E[Y(g;k)]/E[\tau(1)]$, where $Y(g;k) = \sum_{j=T_1(k-1)}^{T_1(k)-1} g(X_j)$ and $\tau(k) = T_1(k) - T_1(k-1)$ for $k \geq 1$. One can show that $E[Y(g;k)^2] < \infty$ and $E[\tau(k)^2] < \infty$, so $n^{1/2}(r_n - r)/\sigma_g \Rightarrow N(0,1)$, as $n \to \infty$, where $\sigma_g^2 = (E[Y(g;1)^2] + r^2 E[\tau(1)^2] - 2rE[Y(g;1)\tau(1)])/E[\tau(1)]$; see Theorem 2.3 of Shedler (1993). Observe that $E[Y(g;1)^2]$, $E[\tau(1)^2]$, and $E[Y(g;1)\tau(1)]$ each have the form in (3), so we can use the estimator in Theorem 1 to estimate these quantities.

In our experiments, we took $\lambda_i = 1/16$ and $\mu_i = 1$ for all $i$. Also, we let $p_0 = 0.20$, $p_1 = 0.50$, $p_2 = 0.15$, $p_3 = 0.10$, $p_4 = 0.05$, and $p_5 = p_6 = p_7 = p_8 = 0$. In each case we ran 1,000 independent replications, with each replication ending with the cycle in progress after $10^6$ transitions. We let $v = (0,0,0,0,0,0,0,0)$ in all cases. In the first part (i.e., the first three rows) of Table 2, we vary the choice for the state $w$ over those states that have exactly 3 customers at one station and none at each of the other stations. In the second part (i.e., the last five rows) of Table 2, we vary the choice for $w$ over those states that have exactly $n_1$ customers at station 1 for $n_1 = 3, 7, 10, 12, 15$, and no customers at any of the other stations. In each replication we computed the standard and permuted estimates of the time-average variance constant, and then we calculated the sample variances of these estimates over the 1,000 independent replication. The second column in the table gives the ratio of the sample variances for the standard and permuted estimates (so larger is better). The difference in computer run times was negligible (less than 0.01 percent), so the ratio of sample variances is essentially also our estimate of the relative efficiency (Glynn and Whitt 1992) of the permuted estimator.

Now we consider the first part of Table 2. Recall that these rows correspond to states $w$ having exactly 3 customers at a particular station $k$, and no customers at any other station, where $k = 1, 2, 3$. Of these three states, it turns out that the steady-state probability of being in the state with the customers at station 1 (resp., 3) is the highest (resp., lowest). Thus, the

| $w$ | Ratio of sample var. |
|---|---|
| $(3, 0, 0, 0, 0, 0, 0, 0)$ | 3.00 |
| $(0, 3, 0, 0, 0, 0, 0, 0)$ | 1.20 |
| $(0, 0, 3, 0, 0, 0, 0, 0)$ | 1.06 |
| $(3, 0, 0, 0, 0, 0, 0, 0)$ | 3.00 |
| $(7, 0, 0, 0, 0, 0, 0, 0)$ | 8.65 |
| $(10, 0, 0, 0, 0, 0, 0, 0)$ | 11.22 |
| $(12, 0, 0, 0, 0, 0, 0, 0)$ | 10.79 |
| $(15, 0, 0, 0, 0, 0, 0, 0)$ | 11.22 |

Table 2: Comparison of estimator sample variances for queueing network model.

first part of Table 2 seems to indicate that the amount of variance reduction increases as the steady-state probability of the state $w$ increases. In the second part of Table 2, recall that the different rows correspond to states in which there is some number of customers at the first station and no customers at any of the other stations. Of these states, the steady-state probabilities of those having many customers at station 1 are higher than those with fewer customers at station 1, and the second part of Table 2 shows that the variance reduction seems to increase for state $w$ having more customers at station 1. The results in the two parts of Table 2 agree with the suggestions given in Calvin and Nakayama (1997,1998a) that one should try to choose a state $w$ that is visited often to maximize the variance reduction. Finally, we see in Table 2 that permuting can result in a significant decrease in the variance (up to about one order of magnitude).

# 8 Conclusions

In this paper we introduced permuted estimators for various performance measures. In particular, we considered product estimators, which could be used for the estimators of the time-average variance, Tin estimators, and derivative estimators. We also considered importance-sampling estimators. We saw that significant variance reduction can be obtained by permuting. In particular, for a large queueing network, we were able to reduce the variance by up to a factor of 11. We also established central limit theorems for some of our estimators, so that confidence intervals can be constructed.

One topic for further investigation is what to do when there are more than two regenerative sequences available. In the empirical results in Section 7, we saw that for certain pairs of sequences, there is a large variance reduction, whereas for other pairs, there is only modest reduction in variance. We are currently studying the issue of constructing estimators when there are more than two regenerative sequences.

# 9 Proof of Theorem 1

Note that the standard estimator can be written as

$$\widehat{\alpha}_m(\vec{X}) = \frac{1}{m}\left[\sum_{k\in J(1;2)} U(k)V(k) + \sum_{k\in H(1;2)} U(k)V(k)\right].$$

If we permute $T_2$-cycles, then the first term on the right-hand side does not change, but the second term does. Using (5) and (6), we get that

$$\begin{aligned}
\sum_{k\in H(1;2)} U(k)V(k) &= \sum_{k\in H(1;2)} \left(U_{12}(k) + \sum_{l\in B_k} U_{22}(l) + U_{21}(k)\right)\left(V_{12}(k) + \sum_{j\in B_k} V_{22}(j) + V_{21}(k)\right) \\
&= \sum_{k\in H(1;2)} \Bigg(U_{12}(k)V_{12}(k) + U_{12}(k)\sum_{j\in B_k} V_{22}(j) + U_{12}(k)V_{21}(k) \\
&+ \sum_{l\in B_k} U_{22}(l)V_{12}(k) + \sum_{l\in B_k} U_{22}(l)\sum_{j\in B_k} V_{22}(j) + \sum_{l\in B_k} U_{22}(l)V_{21}(k) \\
&+ U_{21}(k)V_{12}(k) + U_{21}(k)\sum_{j\in B_k} V_{22}(j) + U_{21}(k)V_{21}(k)\Bigg).
\end{aligned}$$

Let $U'(k)$, $V'(k)$, $U'_{12}(k)$, $U'_{22}(l)$, $U'_{21}(k)$, $V'_{12}(k)$, $V'_{22}(l)$, $V'_{21}(k)$ be the same as the corresponding unprimed variables except that the primed variables are defined for the permuted sample path $\vec{X}'$. We now examine the expectation with respect to a random permutation of $T_2$-cycles of each of the above terms.

Note that

$$E_*\left[\sum_{k\in H(1;2)} U'_{12}(k)V'_{12}(k)\right] = \sum_{k\in H(1;2)} U_{12}(k)V_{12}(k)$$

and

$$E_*\left[\sum_{k\in H(1;2)} U'_{21}(k)V'_{21}(k)\right] = E_*\left[\sum_{k\in H(1;2)} U_{21}(\rho(k))V_{21}(\rho(k))\right] = \sum_{k\in H(1;2)} U_{21}(k)V_{21}(k),$$

where $\rho(k)$ is the index of the $U_{21}$ (and $V_{21}$) variable that follow the $U_{12}(k)$ (and $V_{12}(k)$) variable after a permutation of the $T_2$-cycles. Note that $\rho(k) \neq \psi(k)$ since $U_{12}(k)$ and $U_{21}(\psi(k))$ are always in the same $T_2$-cycle, no matter how the $T_2$-cycles are permuted. Any of the other $h_{12}-1$ indices from $H(1;2)$ are equally likely, however, so that

$$\begin{aligned}
E_*\left[\sum_{k\in H(1;2)} U'_{12}(k)V'_{21}(k)\right] &= \sum_{k\in H(1;2)} U_{12}(k)\, E_*\left[V_{21}(\rho(k))\right] \\
&= \sum_{k\in H(1;2)} U_{12}(k)\frac{1}{h_{12}-1}\left(\sum_{j\in H(1;2)} V_{21}(j) - V_{21}(\psi(k))\right)
\end{aligned}$$

and

$$\begin{aligned}
E_*\left[\sum_{k\in H(1;2)} V'_{12}(k)U'_{21}(k)\right] &= \sum_{k\in H(1;2)} V_{12}(k)\, E_*\left[U_{21}(\rho(k))\right] \\
&= \sum_{k\in H(1;2)} V_{12}(k)\frac{1}{h_{12}-1}\left(\sum_{j\in H(1;2)} U_{21}(j) - U_{21}(\psi(k))\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
&E_*\left[\sum_{k\in H(1;2)}\left(U'_{12}(k)\sum_{j\in B'_k} V'_{22}(j) + U'_{21}(k)\sum_{j\in B_{k'}} V'_{22}(j)\right)\right] \\
&= \sum_{k\in H(1;2)} U_{12}(k)\, E_*\left[\sum_{j\in B'_k} V_{22}(j)\right] + \sum_{k\in H(1;2)} U_{21}(k)\, E_*\left[\sum_{j\in B'_k} V_{22}(j)\right] \\
&= \frac{1}{h_{12}}\sum_{k\in H(1;2)}\left(U_{12}(k)+U_{21}(k)\right)\sum_{j\in J(2;1)} V_{22}(j),
\end{aligned}$$

and similarly

$$\begin{aligned}
&E_*\left[\sum_{k\in H(1;2)}\left(V'_{12}(k)\sum_{j\in B'_k} U'_{22}(j) + V'_{21}(k)\sum_{j\in B'_k} U'_{22}(j)\right)\right] \\
&= \frac{1}{h_{12}}\sum_{k\in H(1;2)}\left(V_{12}(k)+V_{21}(k)\right)\sum_{j\in J(2;1)} U_{22}(j).
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
&E_*\left[\sum_{k\in H(1;2)}\sum_{l\in B'_k} U'_{22}(l)\sum_{j\in B'_k} V'_{22}(j)\right] \\
&= E_*\left[\sum_{k\in H(1;2)}\sum_{l\in J(2;1)} U_{22}(l)1_{\{l\in B'_k\}}\sum_{j\in J(2;1)} V_{22}(j)1_{\{j\in B'_k\}}\right] \\
&= E_*\left[\sum_{k\in H(1;2)}\left(\sum_{l\in J(2;1)} U_{22}(l)V_{22}(l)1_{\{l\in B_k\}} + \sum_{\substack{i,j\in J(2;1)\\ i\neq j}} U_{22}(i)V_{22}(j)1_{\{i\in B'_k,\, j\in B'_k\}}\right)\right].
\end{aligned}$$

Now note that

$$
\begin{aligned}
&E_* \left[ \sum_{k \in H(1;2)} \sum_{l \in J(2;1)} U_{22}(l) V_{22}(l)\, 1_{\{l \in B'_k\}} \right] \\
&\quad = \sum_{l \in J(2;1)} U_{22}(l) V_{22}(l) \sum_{k \in H(1;2)} P_* \left( l \in B'_k \right) \;=\; \sum_{l \in J(2;1)} U_{22}(l) V_{22}(l),
\end{aligned}
$$

where $P_*$ is the probability measure associated with a random uniform permutation of the $T_2$-cycles. Also,

$$
\begin{aligned}
&E_* \left[ \sum_{k \in H(1;2)} \sum_{\substack{i,j \in J(2;1) \\ i \neq j}} U_{22}(i) V_{22}(j)\, 1_{\{i \in B'_k,\, j \in B'_k\}} \right] \\
&\quad = \sum_{\substack{i,j \in J(2;1) \\ i \neq j}} U_{22}(i) V_{22}(j) \sum_{k \in H(1;2)} P_* \left( i \in B'_k,\, j \in B'_k \right) \\
&\quad = \sum_{\substack{i,j \in J(2;1) \\ i \neq j}} U_{22}(i) V_{22}(j) \sum_{k \in H(1;2)} P_* \left( 1 \in B'_k,\, 2 \in B'_k \right) \\
&\quad = \frac{2}{1 + h_{12}} \sum_{\substack{i,j \in J(2;1) \\ i \neq j}} U_{22}(i) V_{22}(j),
\end{aligned}
$$

where the last step follows from Lemma 6 of Calvin and Nakayama (1998a). Putting this all together yields (11).

# Acknowledgments

The work of the first author was supported by the National Science Foundation under grant DMI-9500173. The work of the second author was supported by the National Science Foundation under grant DMI-9624469.

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