2.2.2 (*a*) Show that $L(u)=\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$ is a linear operator:

We have to show that $L(c_1u_1+c_2u_2)=c_1L(u_1)+c_2L(u_2)$. Let's do this:

$$L(c_1u_1+c_2u_2)=\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial}{\partial x}(c_1u_1+c_2u_2)\right]=\frac{\partial}{\partial x}\left[K_0(x)\left(c_1\frac{\partial u_1}{\partial x}+c_2\frac{\partial u_2}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left[c_1K_0(x)\frac{\partial u_1}{\partial x}\right]+\frac{\partial}{\partial x}\left[c_2K_0(x)\frac{\partial u_2}{\partial x}\right]=c_1\underbrace{\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_1}{\partial x}\right]}_{L(u_1)}+c_2\underbrace{\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_2}{\partial x}\right]}_{L(u_2)}$$

$$=c_1L(u_1)+c_2L(u_2)$$

2.2.2 (*b*) Show that $L(u)=\frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right]$ is **not** a linear operator:

We have to show that $L(c_1u_1+c_2u_2)\neq c_1L(u_1)+c_2L(u_2)$. Let's do this:

$$L(c_1u_1+c_2u_2)=\frac{\partial}{\partial x}\left[K_0(x,c_1u_1+c_2u_2)\frac{\partial}{\partial x}(c_1u_1+c_2u_2)\right]$$

$$=\frac{\partial}{\partial x}\left[K_0(x,c_1u_1+c_2u_2)\left(c_1\frac{\partial u_1}{\partial x}+c_2\frac{\partial u_2}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left[c_1K_0(x,c_1u_1+c_2u_2)\frac{\partial u_1}{\partial x}\right]+\frac{\partial}{\partial x}\left[c_2K_0(x,c_1u_1+c_2u_2)\frac{\partial u_2}{\partial x}\right]$$

$$=c_1\underbrace{\frac{\partial}{\partial x}\left[K_0(x,c_1u_1+c_2u_2)\frac{\partial u_1}{\partial x}\right]}_{\neq L(u_1)}+c_2\underbrace{\frac{\partial}{\partial x}\left[K_0(x,c_1u_1+c_2u_2)\frac{\partial u_2}{\partial x}\right]}_{\neq L(u_2)}$$

There is no way to simplify this any further, unless we are given the function $K_0(x,u)$. This does **not** equal $c_1L(u_1)+c_2L(u_2)$

Consider a simple example: $K_0(x,u)=u$. Then $L(u)=\frac{\partial}{\partial x}\left[u\frac{\partial u}{\partial x}\right]$, and from the last line above we get:

$$L(c_1u_1+c_2u_2)=c_1\frac{\partial}{\partial x}\left[(c_1u_1+c_2u_2)\frac{\partial u_1}{\partial x}\right]+c_2\frac{\partial}{\partial x}\left[(c_1u_1+c_2u_2)\frac{\partial u_2}{\partial x}\right]$$

$$=c_1^2\frac{\partial}{\partial x}\left[u_1\frac{\partial u_1}{\partial x}\right]+c_1c_2\frac{\partial}{\partial x}\left[u_2\frac{\partial u_1}{\partial x}\right]+c_2c_1\frac{\partial}{\partial x}\left[u_1\frac{\partial u_2}{\partial x}\right]+c_2^2\frac{\partial}{\partial x}\left[u_2\frac{\partial u_2}{\partial x}\right]$$

$$=\boxed{c_1^2L(u_1)+c_2^2L(u_2)+c_1c_2\left\{\frac{\partial}{\partial x}\left[u_1\frac{\partial u_2}{\partial x}\right]+\frac{\partial}{\partial x}\left[u_2\frac{\partial u_1}{\partial x}\right]\right\}}$$

$$\neq c_1L(u_1)+c_2L(u_2).$$