MATH 331-001
Final Examination
December 14, 2007

1. (24) Solve the Laplace's equation in a half-disk, $0<r<R$, $0<\theta<\pi$:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$
$$|u(0,\theta)|<\infty;\quad u(R,\theta)=T_0\sin\theta$$
$$u(r,0)=u(r,\pi)=0$$

   a) Separate the variables to find the two ODEs
   b) Which of the two ODEs is a boundary value problem? Solve it.
   c) Complete the solution of this PDE, and determine all coefficients. Check that your solution satisfies the given boundary conditions

2. (24) Consider the heat equation for a 1D rod with mixed boundary conditions:

$$\begin{cases} u_t = k u_{xx},\ t>0,\ 0<x<L \\ u(0,t)=0,\ t>0 \\ \dfrac{\partial u}{\partial x}(L,t)=\pm h u(L,t),\ t>0 \ \leftarrow \text{ BC of 3rd kind} \\ u(x,0)=1 \end{cases}$$

   a) Assuming that $h>0$, find the correct sign in the boundary condition at x=L, in order for the boundary condition to make sense physically.
   b) Separate the variables and solve the boundary value part of the problem. Indicate geometrically the eigenvalues using a function graph, and find the asymptotic value of λ_n for large n
   c) Complete the solution of this PDE, and compute the coefficients C_n in terms of λ_n. Finally, use the large-n approximation you found in part (b) to estimate C_n

3. (22) Consider the following Sturm-Liouville problem

$$\begin{cases} \dfrac{d}{dx}\left(x\dfrac{d\phi}{dx}\right)-\dfrac{\phi}{4x}+\lambda x\phi=0 \\ \phi(0)=\phi(1)=0;\quad \phi(x)^2/x \text{ bounded at } x=0 \end{cases}$$

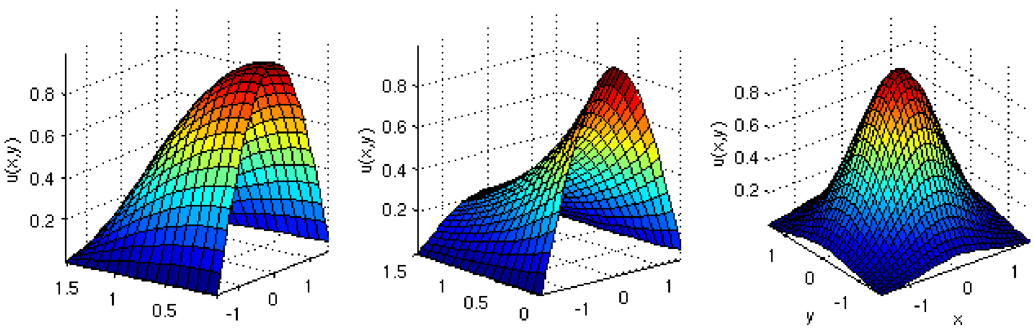
   a) Prove that the eigenvalues are non-negative by deriving the Rayleigh quotient. Write down the orthogonality condition for the eigenfunctions.
   b) Find all eigenfunctions and eigenvalues [hint: multiply the equation by x. If you still can't recall the solution, use the substitution $\varphi(x)=f(\sqrt{\lambda}\,x)/\sqrt{x}$]
   c) Check the orthogonality of the eigenfunctions.

4. (24) Consider the heat equation on an infinite rod with heat loss along its length:

$$\begin{cases} u_t = k u_{xx} - \alpha u, & -\infty < x < +\infty \\ u(x,0) = e^{-x^2/\gamma} \end{cases}$$

a) Write down the ordinary differential equation satisfied by U(ω,t), the Fourier transform of $u(x,t)$
b) Solve this equation to find U(ω,t).
c) Find $u(x,t)$ by taking the inverse transform
d) Check that your answer satisfies the initial condition

5. (10) Which of the following functions satisfies/satisfy the two-dimensional Laplace's equation, $u_{xx} + u_{yy} = 0$? [Hint: note the curvature]



Some facts you may find useful:

$f(x)$	F(ω)
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$
$\sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$
$\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x}) g(x-\bar{x}) d\bar{x}$	F(ω)G(ω)

Bessel equation: $z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2) f = 0$

Large z:

$$J_m(z) \sim \frac{\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)}{\sqrt{z}}$$

$$Y_m(z) \sim \frac{\sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)}{\sqrt{z}}$$

Small z:

$$J_m(z) \sim z^m$$

$$Y_m(z) \sim \begin{cases} \ln z, & m = 0 \\ z^{-m}, & m > 0 \end{cases}$$