1. (10) Consider the heat equation for a 1D rod with **non-constant** thermal properties (i.e. \( c, \rho, K_0 \) are not constant):

\[
c(x)\rho(x) \frac{\partial u}{\partial t} = K_0(x) \frac{\partial^2 u}{\partial x^2} + \ldots + Q(x,t)
\]

a) Write down the missing term “…”

b) Explain (in one sentence) the minus sign in the expression for the heat flux:

\[
\phi = -K_0 \frac{\partial u}{\partial x}
\]

2. (20) Find the equilibrium solution, if it exists. If it does not exist, explain why:

\[
u_t = 2u_{xx} + x^2
\]

\[
u_x(0,t) = 0
\]

\[
u_x(L,t) = 1
\]

\[
u(x,0) = \cos \frac{\pi x}{2L}
\]

a) Find the equilibrium solution as a function of distance from center, \( u_{eq}(\rho) \)

b) Separate the variables and write down the boundary value problem that we would need to consider in order to solve this PDE. **Do not solve.**
4. (50) Solve the Laplace’s equation inside the rectangle $[0, L] \times [0, H]$ with the following boundary conditions.

\[
\begin{align*}
\frac{\partial u}{\partial y}(x, 0) &= \frac{\partial u}{\partial y}(x, H) = 0 \\
u(0, y) &= 1 \\
u(L, y) &= 3\cos \frac{2\pi y}{H}
\end{align*}
\]

a) Sketch the domain and label the boundary conditions. Use the linearity of the Laplace’s equation to break up this problem into manageable parts.

b) Separate the variables, and write down the resulting system of ODEs along with their boundary conditions.

c) Complete the solution; determine all coefficients, and make sure to consider the cases $\lambda = 0, \lambda > 0, \lambda < 0$. Check that your solution satisfies the boundary conditions.

[Hint: It may help to shift the variable by $L$ or $H$ when solving one of the steps]