Math 331-001  *   Final Exam Solution  *  December 19, 2008

\[
\frac{\partial^2 u}{\partial t^2} = 4u, \quad 0 < x < 1
\]

(1) \[
\begin{align*}
\frac{\partial u}{\partial x}(0, t) &= \frac{\partial u}{\partial x}(1, t) = 0 \\
u(x, 0) &= 0, \quad \frac{\partial u}{\partial t}(x, 0) = 5 + 3\cos(2\pi x)
\end{align*}
\]

\[
u(x, t) = f(x)h(t)
\]

\[
f''(t) = h(t) f''(x) - 4f(x)h(t) \div f(x)h(t)
\]

\[
h'' = -\lambda f
\]

\[
f'(0) = f'(1) = 0 \quad \Rightarrow \quad \lambda = 0
\]

\[
h(0) = 0
\]

\[
\lambda = (n\pi)^2 \quad \Rightarrow \quad h(t) = \sin(\sqrt{4 + \lambda} t) \quad \text{"natural" frequencies:} \quad \omega_n = \sqrt{4 + (n\pi)^2}
\]


general solution: \[u(x, t) = f_0(x) h_0(t) + \sum_{n=1}^{\infty} f_n(x)h_n(t) = C_0 \sin(2t) + \sum_{n=1}^{\infty} C_n \cos(n\pi x) \sin(\omega_n t)\]

Now, find the fourier coefficients using the non-homogeneous initial condition:

\[
\frac{\partial u}{\partial t}(x, 0) = 2C_0 + \sum_{n=1}^{\infty} \omega_n C_n \cos(n\pi x) = 5 + 3\cos(2\pi x)
\]

The given initial velocity is already expanded into fourier series, there is nothing to integrate here: simply match the coefficients on the left and on the right:

\[
\begin{align*}
2C_0 &= 5 \\
\omega_2 C_2 &= 3 \\
\text{All other } C_n &= 0
\end{align*}
\]

(2) Check the PDE:

\[
\frac{\partial^2 u}{\partial t^2} = -\frac{5}{2} \sin(2t) - \frac{3}{\omega_2} \omega_2^2 \cos(2\pi x) \sin(\omega_2 t) = -10\sin(2t) - 3\omega_2 \cos(2\pi x) \sin(\omega_2 t)
\]

\[
\frac{\partial^2 u}{\partial x^2} - 4u = -\frac{3}{\omega_2} (2\pi)^2 \cos(2\pi x) \sin(\omega_2 t) - 4\left(\frac{5}{2} \sin(2t) + \frac{3}{\omega_2} \cos(2\pi x) \sin(\omega_2 t)\right)
\]

\[
= -10\sin(2t) - 3\frac{(2\pi)^2 + 4}{\omega_2} \cos(2\pi x) \sin(\omega_2 t) = -10\sin(2t) - 3\omega_2 \cos(2\pi x) \sin(\omega_2 t)
\]
(a) Rayleigh quotient - multiply by $\phi$ and integrate:

$$\int_0^1 \left( x \frac{d\phi}{dx} \right) \phi \, dx = -\lambda \int_0^1 x \phi^2 \, dx \quad \Rightarrow \quad \lambda = \frac{\int_0^1 \left( x \frac{d\phi}{dx} \right)^2 \, dx}{\int_0^1 x \phi^2 \, dx}$$

(b) Simplest test function satisfying these boundary conditions: $\phi(x) = 1 - x^2$

$$\lambda \leq \frac{\int_0^1 (-2x)^2 \, dx}{\int_0^1 \left( x - 2x^3 + x^5 \right) \, dx} = \frac{4 \int_0^1 x^3 \, dx}{\frac{1}{2} - \frac{2}{4} + \frac{1}{6}} = 6 \quad \Rightarrow \quad \lambda \leq 6$$

(c) This S-L problem is the BVP for heat equation or wave equation on a disk (see homework):

- multiply the ODE by $x$ (note: same problem as last year's exam):

$$x \frac{d}{dx} \left( x \frac{d\phi}{dx} \right) = -\lambda x^2 \phi \quad \Rightarrow \quad x^2 \frac{d^2\phi}{dx^2} + x \frac{d\phi}{dx} + \lambda x^2 \phi^2 = 0 \quad \text{Bessel equation of order } m=0$$

Solution: $\phi(z) = AJ_0(z) + BY_0(z)$ Both $Y_0(z)$ and its derivative are unbounded at $z=0$ ($Y_0(z) \sim \ln z$)

Note that $J_0(z)$ satisfies the boundary condition $\phi(0)=0$ since $Y_0(z) \sim 1 - \frac{z^2}{2}$ (see given series expansion)

Boundary condition at $z=1$ gives the eigenvalues: $\phi(x=1) = \phi(z = \sqrt{\lambda}) = J_0(\sqrt{\lambda}) = 0$

Therefore $\sqrt{\lambda_n} = z_{0n} = n$-th zero of $J_0(z)$ and so $\phi_n(x) = J_0(z_{0n} x)$

Asymptotic expression for zeros: $J_0(z) \sim \frac{1}{\sqrt{z}} \cos \left( z - \frac{\pi}{4} / \left( \frac{z}{n-\frac{1}{2}} \right) \right) = 0$ $\Rightarrow z_{0n} = \frac{\pi}{4} + \pi \left( \frac{n}{4} - \frac{1}{2} \right) = \frac{\pi}{2} \left( n - \frac{1}{4} \right) = \sqrt{\lambda_n}$

Note: this asymptotic expression is very accurate even for $n=1$, giving $\lambda_1 \sim (3\pi/4)^2 \sim (9/4)^2 \sim 5 \leq 6$
Since the Fourier transform of $e^{-x^2/4}$ (see provided table, $\alpha=\frac{1}{4}$) is $\frac{1}{\sqrt{4\pi}} e^{-\omega^2}$, we get

$$u(x,t) = \int_{-\infty}^{+\infty} F(\omega)e^{+i\omega t - i\omega x} d\omega = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\omega^2} e^{+i\omega t - i\omega x} d\omega$$

Note that this is an inverse transform of a Gaussian (see table, $\beta=1$), with an argument $(x-\gamma t)$ ("shift theorem")

$$u(x,t) = \int_{-\infty}^{+\infty} e^{-(x-\gamma t)^2/4} = \frac{\sqrt{\gamma t}}{\sqrt{\pi}} e^{-\gamma t^2/4}$$

Sketch: the solution is the same Gaussian as the initial condition, but its center peak is moving to the right with speed $\gamma$.

This equation is a type of a wave equation.

Check the PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \cos x, \quad 0 < x < 1$$

(4)

$$\begin{aligned}
\frac{\partial u}{\partial t} & = \frac{\partial^2 u}{\partial x^2} - \cos x, \quad 0 < x < 1 \\
\frac{\partial u}{\partial x} (0,t) = u(0,t); \quad \frac{\partial u}{\partial x} (1,t) = 0; \quad u(x,0) = f(x)
\end{aligned}$$

(a) the correct sign is positive: temperature is taken out of the left end in proportion to its temperature,

so the heat flux is negative (left-ward): $\phi(0,t) = -\frac{\partial u}{\partial x} (0,t) < 0 \implies \phi(0,t) = -u(0,t) \implies \frac{\partial u}{\partial x} (0,t) = u(0,t)$

(b) Equilibrium temperature distribution:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \cos x = 0 \implies \frac{\partial^2 u}{\partial x^2} = \cos x \implies \frac{\partial u}{\partial x} = \sin x + C_1 \implies u(x,t) = -\cos x + C_1 x + C_2$$

Satisfy the boundary conditions:

$$\begin{aligned}
\frac{\partial u}{\partial x} (1,t) = 0 & \implies \sin 1 + C_1 \implies C_1 = -\sin 1 \\
\frac{\partial u}{\partial x} (0,t) = u(0,t) & \implies \sin 0 + C_1 = -\cos 0 + C_1 \cdot 0 + C_2 \implies C_2 = 1 - \sin 1 \\
\end{aligned}$$

$\implies u(x,t) = -\cos x - (\sin 1)x + 1 - \sin 1$
5. Which of these functions satisfy(ies) the following PDE: \( \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \)

[Hint: simply examine the signs of \( \frac{\partial u}{\partial t} \) and \( \frac{\partial u}{\partial x} \) at a couple different \((x,y)\) points: is the function increasing or decreasing at this point?]

**ANSWER:**

At the magenta point, the function **increases** along the \( x \)-axis and along the \( t \)-axis (and the function increases at the same rate in both direction), so \( \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} > 0 \)

At the green point, the function **decreases** along the \( x \)-axis and along the \( t \)-axis (and the function decreases at the same rate in both direction), so \( \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} < 0 \)

If you examine each of the other three panels, you will easily find point where the function increases along \( x \) but decreases along \( t \), or vice versa.