You can choose between problem 1a and problem 1b (no extra credit for doing both)

(1a, 15pts) The following equation describes the conservation of energy in a thin rod:
\[
\frac{\partial e}{\partial t} = \pm \frac{\partial \varphi}{\partial x}
\]
Here \( e(x,t) \) is the energy density. Derive this equation, and give the correct value of the sign of the right-hand side; sketch a simple picture to explain the derivation. What is the meaning and the physical units of function \( \varphi(x,t) \)?

(1b, 15pts) Solve this ODE to find \( y(x) \) [hint: we solved this type of ODE in class]
\[
\begin{cases}
    x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 3y = 0 \\
y(1) = y'(1) = 0
\end{cases}
\]

(2, 15pts) Separate the variables in the following partial differential equation for \( u(x,y) \) and write down the resulting two ODEs (Do not solve).
\[
\frac{\partial^3 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial y^2} + 3 \frac{\partial u}{\partial y} = 0
\]

(3, 15pts) Find the sine series for the function \( f(x) = x \) on the interval \([0, 1]\) \((L=1)\). Write down the first three non-zero terms. Is this sine series continuous? Explain the answer about the continuity using a sketch.

(4, 15pts) Find the equilibrium solution \((do\ not\ calculate\ the\ full\ time-dependent\ solution)\). [Note the third power of \( r\)]
\[
\begin{cases}
    \frac{\partial u}{\partial t} = k \frac{\partial}{r^3} \frac{\partial}{\partial r} \left(r^3 \frac{\partial u}{\partial r}\right), & 1 \leq r \leq 2 \\
u(1,t) = 0, & u(2,t) = 3 \\
u(r,0) = f(r)
\end{cases}
\]

(5, 40pts) Solve the Laplace’s equation \((u_{xx} + u_{yy}=0)\) with the given boundary conditions inside a rectangle, \(0 \leq x \leq L, \ 0 \leq y \leq H\). Show all the steps in your solution, but you don’t have to explain each step in a lot of detail. Check your answer.
\[
\begin{cases}
    \frac{\partial u}{\partial y}(x,0) = 0, & u(x,H) = 0 \\
u(0,y)=u(L,y) = 4 \cos \frac{\pi y}{2H}
\end{cases}
\]