Math 331, FINAL EXAMINATION, Fall 2006
Wednesday, December 20, 11:30 AM - 2:00 PM

**Problem #1:** The effect of periodic surface heating on the interior temperature of the earth may be modeled by

\[ u_t = \kappa u_{xx}; \quad 0 < x < \infty \]
\[ u(x, 0) = 0; \quad 0 < x < \infty \]
\[ u(0, t) = \cos t; \quad t > 0 \]

Solve for \( u(x, t) \) using an appropriate transform. Leave your answer in the form of an inverse transform integral.

**Problem #2:** Use an appropriate transform to solve

\[ u_{xx} + u_{yy} = 0; \quad 0 < x < L, \ 0 < y < \infty \]

subject to the following boundary conditions

\[ u(0, y) = f(y); \quad 0 < y < \infty \]
\[ u_y (x, 0) = 0; \quad 0 < x < L \]
\[ u(L, y) = 0; \quad 0 < y < \infty \]

Leave your answer in the form of an inverse transform integral.

**Problem #3:** Consider the following Sturm-Liouville Boundary Value Problem:

\[ \frac{d}{dx} (x \phi') + \frac{\lambda}{x} \phi = 0; \quad x \in (1, 2) \]
\[ \phi(1) = 0 \]
\[ \phi(2) = 0 \]
1. Prove that the eigenvalues are real and that they satisfy $\lambda > 0$.

2. Prove that eigenfunctions belonging to different eigenvalues form an orthogonal set of functions over a) what interval? and b) with respect to what weighting function?

3. Find the all the eigenvalues $\lambda_n$ and the corresponding eigenfunctions $\phi_n(x)$.

4. Evaluate $\int_0^1 \phi_n^2(x) \sigma(x) dx$ (the normalization constant). The integral can be evaluated by the substitution $s = \frac{\ln x}{\ln 2}$.

**Problem #4:** Consider Laplace’s equation

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

in the region $1 < r < 2$, $0 < \theta < \pi$, subject to the boundary conditions

- $u(r, 0) = 0$, $u(r, \pi) = 1$; $1 < r < 2$
- $u(1, \theta) = 0$, $u(2, \theta) = 0$; $0 < \theta < \pi$

Apply the method of separation of variables, and the results from problem #3 to find $u(r, \theta)$. At some point, the substitution $r = e^s$ may come in handy.

**Useful formulas and Fourier/Sine/Cosine transforms**

**Generalities on Sturm-Liouville Boundary Value Problems**

\[
(p(x)\phi'(x))' + q(x)\phi + \lambda \sigma(x)\phi(x) = 0; \quad x \in (a, b)
\]

\[
\alpha_1\phi(a) + \alpha_2\phi'(a) = 0; \quad B.C. \ at \ x = a
\]

\[
\alpha_3\phi(b) + \alpha_4\phi'(b) = 0; \quad B.C. \ at \ x = b,
\]

where $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ are real constants. When (1) is recast in the form $L\phi = \lambda \sigma(x)\phi$, we have Green’s second and first formulas for the operator $L$ over $[a, b]$ as follows:

\[
\int_a^b ((Lf)g - f(Lg))dx = -[p(x)(f'(x)g(x) - f(x)g'(x))]_{x=a}^b,
\]

\[
\int_a^b (Lf)gdx = \int_a^b p(x)f'(x)g'(x)dx - \int_a^b q(x)f(x)g(x)dx - [p(x)f(x)g(x)]_{x=a}^b,
\]
where $f(x)$ and $g(x)$ are two functions with at least 2 derivatives.