Math 331, **FINAL EXAMINATION**, Fall 2006
Wednesday, December 20, 11:30 AM - 2:00 PM

Problem #1: The effect of periodic surface heating on the interior temperature of the earth may be modeled by

$$u_t = \kappa u_{xx}; \quad 0 < x < \infty$$

$$u(x,0) = 0; \quad 0 < x < \infty$$

$$u(0,t) = \cos t; \quad t > 0$$

Solve for $u(x,t)$ using an appropriate transform. Leave your answer in the form of an inverse transform integral.

Problem #2: Use an appropriate transform to solve

$$u_{xx} + u_{yy} = 0; \quad 0 < x < L, \ 0 < y < \infty$$

subject to the following boundary conditions

$$u(0,y) = f(y); \quad 0 < y < \infty$$

$$u_y(x,0) = 0; \quad 0 < x < L$$

$$u(L,y) = 0; \quad 0 < y < \infty$$

Leave your answer in the form of an inverse transform integral.

Problem #3: Consider the following Sturm-Liouville Boundary Value Problem:

$$\frac{d}{dx}(x\phi') + \frac{\lambda}{x}\phi = 0 \ ; \quad x \in (1,2)$$

$$\phi(1) = 0$$

$$\phi(2) = 0$$

1. Prove that the eigenvalues are real and that they satisfy $\lambda > 0$.

2. Prove that eigenfunctions belonging to different eigenvalues form an orthogonal set of functions over a) what interval ? and b) with respect to what weighting function ?

3. Find the *all* the eigenvalues λ_n and the corresponding eigenfunctions $\phi_n(x)$.

4. Evaluate $\int_1^2 \phi_n^2(x)\sigma(x)dx$ (the normalization constant). The integral can be evaluated by the substitution $s = \pi\frac{\ln x}{\ln 2}$.

Problem #4: Consider Laplace's equation

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in the region $1 < r < 2,\ 0 < \theta < \pi$, subject to the boundary conditions

$$u(r,0) = 0, \quad u(r,\pi) = 1; \quad 1 < r < 2$$
$$u(1,\theta) = 0, \quad u(2,\theta) = 0; \quad 0 < \theta < \pi$$

Apply the method of separation of variables, and the results from problem #3 to find $u(r,\theta)$. At some point, the substitution $r = e^s$ may come in handy.

<u>Useful formulas and Fourier/Sine/Cosine transforms</u>

Generalities on Sturm-Liouville Boundary Value Problems

$$\begin{aligned} (p(x)\phi'(x))' + q(x)\phi + \lambda\sigma(x)\phi(x) &= 0; \quad x \in (a,b) \qquad (1) \\ \alpha_1\phi(a) + \alpha_2\phi'(a) &= 0; \quad B.C. \ at \ x = a \\ \alpha_3\phi(b) + \alpha_4\phi'(b) &= 0; \quad B.C. \ at \ x = b, \end{aligned}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$ are real constants. When (1) is recast in the form $\mathcal{L}\phi = \lambda\sigma(x)\phi$, we have Green's second and first formulas for the operator $\mathcal{L}$ over $[a,b]$ as follows:

$$\int_a^b ((\mathcal{L}f)g - f(\mathcal{L}g))dx = -[p(x)(f'(x)g(x) - f(x)g'(x))]_{x=a}^{x=b},$$

$$\int_a^b (\mathcal{L}f)g dx = \int_a^b p(x)f'(x)g'(x)dx - \int_a^b q(x)f(x)g(x)dx - [p(x)f'(x)g(x)]_{x=a}^{x=b},$$

where $f(x)$ and $g(x)$ are two functions with at least 2 derivatives.