MATH 331-001
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Midterm Examination #1
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Calculators not allowed. You must remain seated until you hand in the solution. Please read each problem carefully, and show all work.

1. (10) Consider the heat equation for a 1D rod with **non-constant** thermal properties (i.e. c, ρ, K_0 are not constant):

$$c(x)\rho(x)\frac{\partial u}{\partial t} = K_0(x)\frac{\partial^2 u}{\partial x^2} + \ldots + Q(x,t)$$

   a) Write down the missing term "…"
   b) Explain (in one sentence) the minus sign in the expression for the heat flux:
   $\phi = -K_0 \frac{\partial u}{\partial x}$

2. (20) Find the equilibrium solution, if it exists. If it does not exist, explain why:

a) $$\begin{cases} u_t = 2u_{xx} + x^2 \\ \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(L,t) = 1 \\ u(x,0) = \cos\frac{\pi x}{2L} \end{cases}$$

b) $$\begin{cases} u_t = u_{xx} - e^x \\ \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(1,\ t) = e - 1 \\ u(x,0) = e^x - x^2/2 \end{cases}$$

3. (20) Consider the heat equation for the symmetric temperature distribution $u(\rho,t)$ inside a sphere of radius R_1 (ρ is the distance from the center). The sphere is cooled at the surface, and has a core of radius R_0 heated to 20 degrees:

$$\begin{cases} u_t = \frac{k}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) & \leftarrow \text{Laplacian for spherically symmetric case} \\ u(R_0,t) = 20 \\ u(R_1,t) = 0 \\ u(\rho,0) = 20\frac{R_1 - \rho}{R_1 - R_0} \end{cases}$$

   a) Find the equilibrium temperature as a function of distance from center, $u_{eq}(\rho)$

   b) Separate the variables and write down the boundary value problem that we would need to consider in order to solve this PDE. **Do not solve**.

4. (50) Solve the Laplace's equation inside the rectangle [0, L] × [0, H] with the following boundary conditions.

$$\begin{cases} \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,H) = 0 \\ u(0,y) = 1 \\ u(L,y) = 3\cos\frac{2\pi y}{H} \end{cases}$$

a) Sketch the domain and label the boundary conditions. Use the linearity of the Laplace's equation to break up this problem into manageable parts.

b) Separate the variables, and write down the resulting system of ODEs along with their boundary conditions.

c) Complete the solution; determine all coefficients, and make sure to consider the cases $\lambda=0$, $\lambda>0$, $\lambda<0$. Check that your solution satisfies the boundary conditions.

[Hint: It may help to shift the variable by L or H when solving one of the steps]