Math 331-001 * Final Examination * December 14, 2010
This is a closed-book test: notes and calculators are *not* permitted.

1. (25pts) Solve the following Laplace's equation in a rectangle. Determine all coefficients and simplify the solution as much as possible.

$$\begin{cases} \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < 1,\ 0 < y < 2) \\ \dfrac{\partial u}{\partial x}(0, y) = \dfrac{\partial u}{\partial x}(1, y) = 0 \\ u(x,0) = x,\ u(x,2) = 0 \end{cases}$$

2. (30pts) Solve the wave equation for a drum shaped as a circular sector, with free side edges and fixed outer edge. Simplify the final answer as much as possible

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left\{ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial u}{\partial r} \right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} \right\} \quad \left(0 < r < 1,\ 0 < \theta < \dfrac{\pi}{4},\ t > 0 \right) \\ \dfrac{\partial u}{\partial \theta}(r,0,t) = \dfrac{\partial u}{\partial \theta}\left(r, \dfrac{\pi}{4}, t \right) = 0 \\ u(0,\theta,t) \text{ is bounded; } \ u(1,\theta,t) = 0 \\ u(r,\theta,0) = 0,\ \dfrac{\partial u}{\partial t}(r,\theta,0) = \sin(\pi r) \cos(8\theta) \end{cases}$$

3. (20pts) Consider the following boundary value problem:

$$\begin{cases} \dfrac{d}{dx}\left(x \dfrac{d\phi}{dx} \right) - 2\phi(x) + \lambda \phi(x) = 0, \qquad 0 < x < \pi \\ \dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(\pi) = 0 \end{cases}$$

a) Find the Rayleigh quotient for this problem
b) Use the Rayleigh quotient to determine whether any eigenvalue can be less than or equal to zero.
c) Use *the simplest* test function to find an upper bound on the lowest λ (using Raleigh quotient)

Choose between problems 4 and 4'

4. (15pts) Find the first two non-zero terms in the Taylor series for *f*(*z*) (hint: plug in a power series for *f*(*z*), and find its coefficients by equating two sides of the equation). Make a rough sketch of *f*(*z*)

$$\begin{cases} z^2 \dfrac{d^2 f}{dz^2} + z \dfrac{df}{dz} + (z^2 - 1) f = 0 \\ f(0) = 0,\ \dfrac{df}{dz}(0) = 1 \end{cases}$$

4' (15pts) Solve the following boundary value problem (consider λ<0 and λ>0); sketch the first two eigenfunctions.

$$\begin{cases} \dfrac{d^2 f}{dx^2} + \lambda f(x) = 0 \qquad (0 < x < 2) \\ \dfrac{df}{dx}(0) = 0, \quad \dfrac{df}{dx}(2) = -f(2) \end{cases}$$

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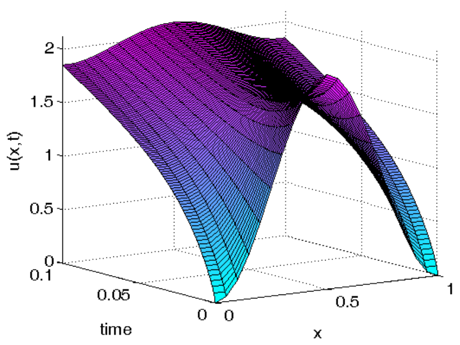
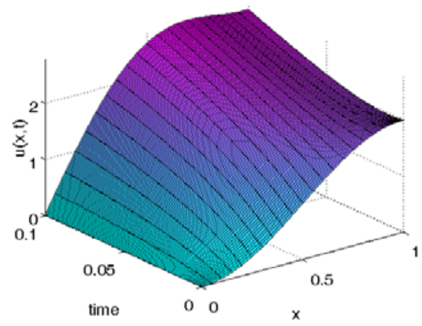
5. (10pts) Which of the following surfaces represents the solution to the following equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \sin \pi x \quad (0 < x < 1,\ t > 0) \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0 \\ u(x,0) = 1 - \cos(2\pi x) \end{cases}$$

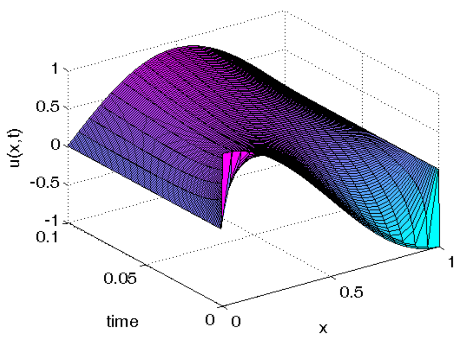
(a)



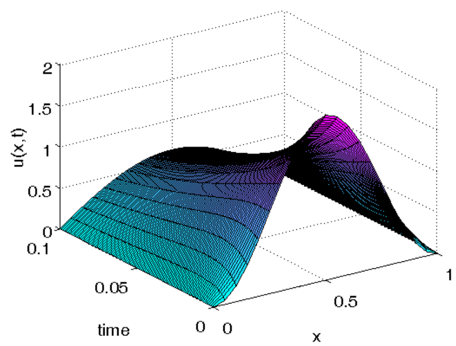
(b)



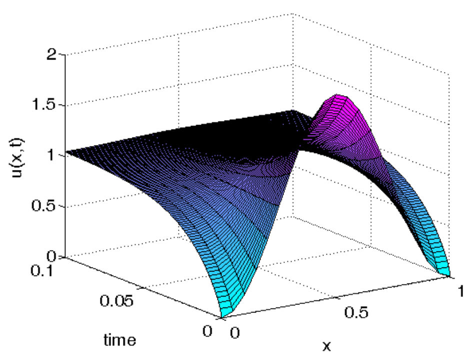
(c)



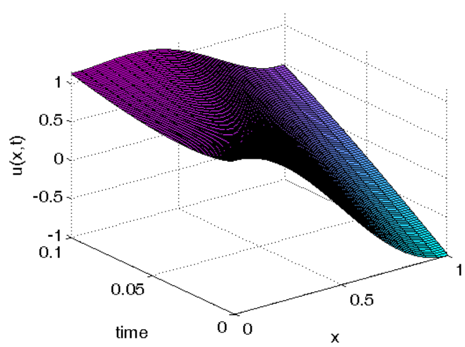
(d)



(e)



(f)



Some facts you may (or may not) find useful:

Bessel equation: $z^2 \dfrac{d^2 f}{dz^2} + z \dfrac{df}{dz} + (z^2 - m^2) f = 0$

Large-z asymptotics: $J_m(z) \sim \dfrac{1}{\sqrt{z}} \cos\left(z - \dfrac{\pi}{4} - m\dfrac{\pi}{2} \right)$, $\quad Y_m(z) \sim \dfrac{1}{\sqrt{z}} \sin\left(z - \dfrac{\pi}{4} - m\dfrac{\pi}{2} \right)$