MATH 331-001 * Midterm Examination * October 12, 2010

1. **(12pts)** Choose the correct sign in each of the following boundary conditions for the temperature of a rod of length L. Here u_o is the temperature of the outside material in contact with the ends of the rod. Explain your choice of sign (**consider the direction of heat flux**):

$$\frac{\partial u}{\partial x}(L,t) = \pm\gamma\left[u(L,t) - u_o\right]; \quad \frac{\partial u}{\partial x}(0,t) = \pm\gamma\left[u(0,t) - u_o\right] \quad \text{(Here } \boldsymbol{\gamma > 0}\text{)}$$

2. **(16pts)** Find the **first four** non-zero terms in the sine series for the function $f(x) = \begin{cases} 1, & \frac{1}{2} < x < \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}$ on the interval [0, 1]. Sketch what the **full** sine series would look like, graphed on the interval [−1, 2].

3. **(20pts)** Consider the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - 4u \quad (0 < x < 1, \ \ t > 0) \\ u(0,t) = 0, \quad \dfrac{\partial u}{\partial x}(1,t) = \pm\mathbf{1} \\ u(x,0) = x \end{cases}$$

(a) For which value of sign in the boundary condition at x=1 does the **equilibrium** temperature distribution exists? Find and sketch the equilibrium temperature distribution, and explain the energy balance in terms of energy entering and leaving the rod

(b) Does the heat energy of the rod/cable remain constant over time? Explain your answer.

4. **(52pts)** Solve the Laplace's equation in a ring sector by following the steps below:

$$\begin{cases} \nabla^2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} \quad \left(\mathbf{0 < \theta < \dfrac{\pi}{3}, \ 1 < r < 2}\right) \\ u(r,0) = \dfrac{\partial u}{\partial \theta}\left(r, \dfrac{\pi}{3}\right) = 0 \\ u(1,\theta) = 0 \\ u(2,\theta) = \sqrt{2}\sin(3\theta/2) \end{cases}$$

i) (6pts) Sketch the domain and label the boundary conditions. What physical problem does this equation describe (name one)?

ii) (10pts) Separate the variables, and write down the resulting two ODEs along with their boundary conditions.

iii) (10pts) Solve the boundary value problem, and make sure to examine whether there is a zero eigenvalue. Make a graph of any two eigenfunctions.

iv)(10pts) Solve the second ordinary differential equation (note that r=0 is not inside the domain). Combine your solutions and write the general form of $u(r,\theta)$

v) (10pts) Determine all coefficients using the boundary conditions

vi) (6pts) Check your answer: check the equation and all boundary conditions. If the solution does not work, say so explicitly (do this even if you did not complete part "v")