Math 331-001 * Final Examination * December 21, 2016
This is a closed-book test: notes and calculators are *not* permitted.

1. **(14pts)** Suppose $f(x)$=**1** on the interval $0 < x < 1$ (***L*=1**), and is an **odd periodic extension** of this function to the rest of the real line. Find the **sine** series for this odd periodic extension. Make sure to simplify as much as possible. Sketch this odd periodic extension, as well as the first non-zero Fourier term, for $-2<x<2$.

2. **(14pts)** Consider the following heat equation for a rod/cable of length L=1 (here temperature is defined relative to the external heat bath temperature, $u_{Bath}=0$):

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + 12(x-1)^2 & (0<x<1,\ t>0) \\ u_x(0,t) = 2u(0,t) \\ u(1,t) = 0 \end{cases}$$

   a) Determine the equilibrium temperature distribution. Make sure to double-check boundary conditions.
   b) Make a rough sketch of equilibrium solution (you don't have to know the solution to plot it).
   c) Explain the equilibrium: where does the energy enter the rod, and where does it leave the rod?

3. **(18pts)** Solve the following Laplace's equation in a rectangle. Start by solving the BVP with *homogeneous* boundary conditions. Pay particular attention to any zero eigenvalue(s). Determine all coefficients and simplify the solution. Check that your solution satisfies the boundary conditions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & (0<x<L,\ 0<y<H) \\ u(0,y) = 0,\ u(L,y) = 1 \\ \dfrac{\partial u}{\partial y}(x,0) = \dfrac{\partial u}{\partial y}(x,H) = 0 \end{cases}$$

4. **(18pts)** Consider the following boundary value problem:

$$\begin{cases} \dfrac{d}{dx}\left(x\dfrac{d\phi}{dx}\right) - \dfrac{\phi(x)}{x} + \lambda\, x\,\phi(x) = 0, & 0<x<1 \\ \phi(0) = \phi(1) = 0 \end{cases}$$

   a) Find the Rayleigh quotient for this problem. Explain whether $\lambda=0$ is a possible solution.
   b) Find constants b and c so that the simplest quadratic test function $u_T(x) = x^2 + bx + c$ satisfies given boundary conditions. Sketch this test function.
   c) Use this test function to find the upper bound on the lowest eigenvalue, λ_1.
   d) Write down all solutions of this BVP in terms of some Bessel function(s).
   e) Make a rough sketch of the first 3 eigenfunctions on the interval [0, 1].

<u>Choose two out of the last four problems:</u>

5. **(18pts)** We learned that the heat equation $u_t = k\,u_{xx}$ can be solved numerically one time step at a time, using the forward Euler iteration $u_n^{(m+1)} = (1-2s)\,u_n^{(m)} + s\left(u_{n-1}^{(m)} + u_{n+1}^{(m)}\right)$. What does $u_n^{(m)}$ represent? What is parameter s equal to? Explain how to obtain this formula from the finite difference approximations of u_t and u_{xx}. Finally, explain how this iterative formula would change if the heat equation included a heat source or sink term, for instance $u_t = k\,u_{xx} - 2u$

6. **(18pts)** Solve the Laplace's equation in a quarter-disk, $0 < r < R$, $0 < \theta < \pi/4$; make sure to calculate all coefficients:

$$\begin{cases} \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \\ |u(0,\theta)| < \infty,\ u(R,\theta) = 3\cos 8\theta \\ \frac{\partial u}{\partial \theta}(r,0) = \frac{\partial u}{\partial \theta}\left(r,\frac{\pi}{4}\right) = 0 \end{cases}$$

7. **(18pts)** Solve the heat equation below on a rectangular plate. Note that the two BVPs are almost identical. Make sure to calculate all coefficients. What is the largest term in the solution when $t >> 0$?

$$\begin{cases} \frac{\partial u}{\partial t} = k\nabla^2 u \quad \text{where } \nabla^2 u = u_{xx} + u_{yy} \quad (0 < x < 1,\ 0 < y < 2,\ t > 0) \\ \frac{\partial u}{\partial x}(0,y,t) = u(1,y,t) = 0; \quad \frac{\partial u}{\partial y}(x,0,t) = u(x,2,t) = 0 \\ u(x,y,0) = 1 \end{cases}$$

8. **(18pts)** Consider the Laplace's equation on an infinite half-plane:

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad -\infty < x < +\infty,\ 0 < y < +\infty \\ u(x,0) = e^{-x^2/4} \\ u(x, y \to \infty) = 0 \end{cases}$$

a) Write down the ordinary differential equation for $U(\omega, y)$, the Fourier transform of $u(x, y)$

b) Solve this equation to find $U(\omega, y)$.

c) Write down $u(x, y)$ in the form of an inverse Fourier transform integral of $U(\omega, y)$. You don't have to take this final integral.

Some facts you may (or may not) find useful:

Bessel equation: $z^2 f_{zz} + z f_z + (z^2 - m^2) f(z) = 0$

Large-z asymptotics: $J_m(z) \sim \frac{1}{\sqrt{z}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$, $\quad Y_m(z) \sim \frac{1}{\sqrt{z}}\sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$

$f(x) = \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega x} d\omega$	$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$
$\sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$
$\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x}) g(x - \bar{x}) d\bar{x}$	$F(\omega)G(\omega)$
$\frac{\partial f}{\partial x}$	$(-i\omega)\, F(\omega)$