Math 331 * Final Examination * May 13, 2019

Read carefully. This is a closed-book exam. **You may drop one 17-point problem.**

1. (22pts) Solve the Laplace's equation in a disk sector. Make sure to determine all Fourier coefficients:

$$\begin{cases} \nabla^2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 \quad \left(0 < r < R;\ 0 < \theta < \dfrac{\pi}{6}\right) \\ u(r,\theta) \text{ is bounded within the given domain} \\ u(R,\theta) = 3 + 2\cos 6\theta \\ \dfrac{\partial u}{\partial \theta}(r,0) = \dfrac{\partial u}{\partial \theta}\left(r,\dfrac{\pi}{6}\right) = 0 \end{cases}$$

2. (22pts) Solve the following Laplace's equation in a box. Make sure to determine all Fourier coefficients, and write down the first three non-zero terms in the solution, $u(x, y)$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad (0 < x < 1,\ 0 < y < 1) \\ \dfrac{\partial u}{\partial x}(0,y) = 0;\ \ u(1,y) = 1 \\ u(x,0) = 0;\ \ \dfrac{\partial u}{\partial y}(x,1) = 0 \end{cases}$$

3. (22pts) Consider the following regular Sturm-Liouville problem: $\begin{cases} \dfrac{d}{dx}\left[e^x \dfrac{d\phi}{dx}\right] - x\phi + \lambda x^2 \phi = 0 \ \left(0 < x < 1\right) \\ \dfrac{d\phi}{dx}(0) = 0;\ \ \dfrac{d\phi}{dx}(1) = 3\phi(1) \end{cases}$

a) Prove that the eigenfunctions (i.e. the solutions) are orthogonal (hint: cross-multiply, subtract, integrate).

b) Make a rough qualitative plot of the first and the third eigenfunctions.

c) Derive the Rayleigh Quotient for this problem. Can one rule out $\lambda < 0$?

d) Find any simple polynomial test function satisfying these boundary conditions (e.g., a parabola). Plug it into the Rayleigh Quotient, but you **don't have to** complete the integration and the RQ calculation.

<u>You may drop one of the remaining 3 problems:</u>

4. (17pts) Solve the following ordinary differential equations. For the first problem (part "*a*"), make a rough qualitative plot of the solution for $0 < x < 8$

$$(a)\ \begin{cases} x^2 \dfrac{d^2 f}{dx^2} + x\dfrac{df}{dx} + x^2 f(x) = 0 \ \ (x > 0) \\ f(0) = 2 \end{cases} \qquad (\text{b})\ \begin{cases} x^2 \dfrac{d^2 f}{dx^2} + 4x\dfrac{df}{dx} - 4f(x) = 0 \ \ (0 < x < 1) \\ f(0) = 0;\ \ f(1) = 3 \end{cases}$$

5. **(17pts)** Find the **equilibrium** solution for the following heat equation for a cable of length L=1 with constant thermal properties, with a sink term proportional to the temperature (see equation). Then, plot the equilibrium temperature on the interval 0<x<1, and explain the heat balance: where does the energy enter, and where does it leave the cable? Hint: you can assume that $u(x,t)$>0 everywhere within the domain.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - 9u(x,t) \quad (0 < x < 1,\ t > 0) \\ \dfrac{\partial u}{\partial x}(0,t) = -3; \quad u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

6. **(17pts)** Derive the stencil of the modified forward Euler method for the following heat equation with an extra "advection" (transport) term, thermal diffusivity of k=1, and length of L=**2**:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial u}{\partial x} \quad (0 < x < \mathbf{2},\ t > 0) \\ u(0,t) = u(2,t) = \mathbf{1} \end{cases}$$

Use the centered difference approximation $\dfrac{\partial u}{\partial x}(x_n, t_m) \approx \dfrac{u_{n+1}^{(m)} - u_{n-1}^{(m)}}{2\Delta x}$ for the advection term. For the other two derivatives, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^2 u}{\partial x^2}$, use the same finite difference approximations that we used in class and in the homework (forward difference for the time derivative, and centered difference for the spatial derivative). Finally, calculate the solution over one time step, using $\Delta t = 0.02$ and N=5 spatial intervals, with initial condition $u_{n=0..5}^{(0)} = (1, 1, 0, 0, 1, 1)$. To simplify calculations, use quotients instead of decimals.

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Some facts that you may or may not find useful:

Bessel equation of order m: $z^2 \dfrac{d^2 f}{dz^2} + z\dfrac{df}{dz} + (z^2 - m^2) f(z) = 0$

Solution: $f(z) = C_1 J_m(z) + C_2 Y_m(z)$

J_m -- Bessel function of the first kind of order m

Y_m -- Bessel function of the second kind of order m

Small-z asymptotics: $J_m(z) \sim \begin{cases} 1 & (m=0) \\ \dfrac{1}{2^m m!} z^m & (m>0) \end{cases}$ $\quad Y_m(z) \sim \begin{cases} \dfrac{2}{\pi}\ln z & (m=0) \\ -\dfrac{2^m (m-1)!}{\pi} z^{-m} & (m>0) \end{cases}$

Large-z asymptotics: $J_m(z) \sim \sqrt{\dfrac{2}{\pi z}} \cos\left(z - \dfrac{\pi}{4} - m\dfrac{\pi}{2}\right), \quad Y_m(z) \sim \sqrt{\dfrac{2}{\pi z}} \sin\left(z - \dfrac{\pi}{4} - m\dfrac{\pi}{2}\right)$