Math 331 * Spring 2019 * Fourier Transform * Victor Matveev

Underscore character (_) means *y* or *t* or another variable which is not transformed

Inverse Transform:	*Forward Transform:*
$u(x,y)=\int_{-\infty}^{+\infty} U(\omega,y)e^{-i\omega x}d\omega$	$U(\omega,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,y)e^{i\omega x}dx$
$u(x,t)=\int_{-\infty}^{+\infty} U(\omega,t)e^{-i\omega x}d\omega$	$U(\omega,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,t)e^{i\omega x}dx$
$\frac{\partial^n u}{\partial x^n}(x,\_)$	$(-i\omega)^n U(\omega,\_)$
$(ix)^n u(x,\_)$	$\frac{\partial^n U}{\partial \omega^n}(\omega,\_)$
$\frac{\partial^n u}{\partial t^n}$ or $\frac{\partial^n u}{\partial y^n}$	$\frac{\partial^n U}{\partial t^n}$ or $\frac{\partial^n U}{\partial y^n}$
$u(x-x_o,\_)$	$U(\omega,\_)e^{i\omega x_o}$ (***shifting theorem***)
$\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})g(x-\bar{x})d\bar{x}$ (**convolution theorem**)	$F(\omega)G(\omega)$
$\delta(x-x_o)$	$\frac{1}{2\pi}e^{i\omega x_o}$
$2\pi\delta(x)$	1
1	$\delta(\omega)$
$\exp(-\alpha x^2)$	$\frac{1}{\sqrt{4\pi\alpha}}\exp\left(-\frac{\omega^2}{4\alpha}\right)$
$\sqrt{\frac{\pi}{\beta}}\exp\left(-\frac{x^2}{4\beta}\right)$	$\exp(-\beta\omega^2)$
$\frac{2\alpha}{x^2+\alpha^2}$	$\exp(-\alpha\|\omega\|)$
$f(x)=\begin{cases}0, & \|x\|>a \\ 1, & \|x\|<a\end{cases}$	$F(\omega)=\frac{\sin a\omega}{\pi\omega}$

The next page demonstrates direct analogy between the Fourier Transform method and the Fourier Series / separation of variables method.

Solution to heat equation on a finite domain with periodic boundary conditions using Fourier Series (separation of variables):	**Solution to heat equation on an infinite domain using Fourier Transform:**
$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial t^2} \quad \begin{pmatrix} -L < x < L \\ t > 0 \end{pmatrix} \\ u(x,0) = f(x) \\ \text{Periodic boundary conditions at } x = \pm L \end{cases}$ $\Downarrow$ $\dfrac{1}{k}\dfrac{h_n'}{h_n(t)} = \dfrac{\phi_n''(x)}{\phi_n(x)} = -\lambda_n \equiv -\omega_n^2$ $\Downarrow$ $\dfrac{dh_n}{dt} = -k\omega_n^2 h_n(t) \;\Rightarrow\; h_n(t) = \exp(-k\omega_n^2 t)$ $\dfrac{d^2\phi_n}{dx^2} = -\omega_n^2\phi_n(x) \;\Rightarrow\; \phi_n(t) = \exp(\pm i\,\omega_n x)$ $\Downarrow$ $u(x,t) = \sum_{n=-\infty}^{+\infty} C_n \underbrace{\exp(-k\omega_n^2 t)}_{h_n(t)} \underbrace{\exp(-i\omega_n x)}_{\phi_n(x)}$ where (for periodic b.c.'s): $\omega_n = \sqrt{\lambda_n} = \dfrac{n\pi}{L}$ $C_n = \dfrac{1}{2L}\int_{-L}^{+L} f(x) \underbrace{\exp(+i\omega_n x)}_{\overline{\phi_n(x)}} dx$	$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial t^2} \quad \begin{pmatrix} -\infty < x < \infty \\ t > 0 \end{pmatrix} \\ u(x,0) = f(x) \\ \text{Solution vanishes as } x \to \pm\infty \end{cases}$ $\Downarrow$ $\begin{cases} \dfrac{\partial U}{\partial t} = -k\omega^2 U \quad (\text{analogous to } h(t)) \quad \begin{pmatrix} -\infty < \omega < \infty \\ t > 0 \end{pmatrix} \\ U(\omega,0) = F(\omega) \end{cases}$ $\Downarrow$ $U(\omega,t) = \underbrace{F(\omega)}_{\text{Fourier "coefficient"}} \underbrace{\exp(-k\omega^2 t)}_{h_\omega(t)}$ $\Downarrow$ $u(x,t) = \int_{-\infty}^{+\infty} \underbrace{\underbrace{F(\omega)}_{\text{Fourier "coefficient"}} \underbrace{\exp(-k\omega^2 t)}_{h_\omega(t)}}_{U(\omega,t)} \underbrace{\exp(-i\omega x)}_{\phi_\omega(x)} d\omega$ $F(\omega) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty} f(x) \underbrace{\exp(+i\omega x)}_{\overline{\phi_\omega(x)}} dx$