Math 613 * Fall 2018 * Midterm Examination * Victor Matveev

1. **(25pts)** Consider the following ODE (here ε<1 Is a positive constant): $\begin{cases} \dfrac{dy}{dt} = \ln(1-2\varepsilon y) \\ y(0)=1 \end{cases}$

   a) **(4pts)** Find the equilibrium and analyze its stability.

   b) **(6pts)** Make the phase plot for this ODE (d*y*/d*t* vs *y*), for the case *ε*=1/4. Use your phase plot to determine and plot the qualitative behavior of the solution *y*(*t*) vs *t*, without solving this equation.

   c) **(15pts)** Find the first three terms in the asymptotic approximation to the solution for the case ε<<1. Hint: substitute $y(t) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^3)$. Initial conditions are satisfied by setting $y_0(0)=1$, $y_1(0) = y_2(0) = 0$. Another hint: $\ln(1+z) \approx z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - ...$

2. **(20pts)** Non-dimensionalize the following equation for the vertical displacement *y*(*x*,*t*) of a vibrating string, reducing the number of parameters as much as possible:

$$\rho \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} = F_T \frac{\partial^2 y}{\partial x^2} \quad (0 < x < \mathbf{L},\ t > 0)$$

   Here *y* is the vertical displacement (units of length), *t* is time, *x* is the horizontal position along the string, *L* is the length of the string, ρ is linear mass density (mass per length), F_T=const>0 is the string tension force, and γ is the damping (dissipation/friction) coefficient. Start by determining the units of γ.

3. **(20pts)** Consider the following 2D flow:

$$\begin{cases} \dfrac{dx}{dt} = x^2 - y - 2 \\ \dfrac{dy}{dt} = x - y \end{cases}$$

   a) **(10pts)** Find all equilibria of this system, and analyze their stability using linear stability analysis.

   b) **(5pts)** Sketch the nullclines.

   c) **(5pts)** Plot of the flow field. Hint: start by showing the flow along the coordinate axes and the nullclines.

4. **(20pts)** Consider the following advection equation (Burger's equation). Note that the equation is already non-dimensionalized, and that the initial position is constrained to the range [–2, 2].

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho^2) = 0 \quad (t > 0) \\ \rho(x,0) = \rho_o(x) = x^2/8 \quad (-2 \le x \le +2) \end{cases}$$

   a) Find and plot the characteristics corresponding to 5 values of x_0: x_0 =–2, x_0 =–1, x_0 =0, x_0 =1, x_0 =2.

   b) Is there a shock-wave / break-up? Is there a value of *x* for which the solution is constant?

   c) Find the solution by inverting the expression for the characteristics: $x_o = x_0(x,t)$. Note that you will get two solutions: explain why, briefly (hint: see part "a").

   d) Make a *rough* plot of the solution *x*(*t*) at *t*=1/2.

One last problem on the reverse side (choose one of two):

Choose one of the last two problems:

5. **(15pts)** Examine the following catalytic reaction: $A + C \underset{k_1^-}{\overset{k_1^+}{\rightleftharpoons}} E \underset{k_2^-}{\overset{k_2^+}{\rightleftharpoons}} B + C$

   a) Write down the system of differential equations describing reactant concentrations *A*, *B*, *C* and *E*. Hint: be careful with equations on catalyst *C*: note that it appears on both ends of this chemical chain.

   b) Write down two conservation laws in this system (i.e. find two linear combinations of the variables which remain constant over time). Note that you can answer this question just by looking at the reaction diagram, which may help you find any mistakes in your answer to part “a”.

6. **(15pts)** Write down the derivation of the diffusion equation in a one-dimensional domain (tube of length *L*), for the case of non-constant cross-section *A*(*x*). Show all steps in the derivation, and define any variables that you use, including their physical units of measurement.