**Math 613 * Fall 2019 * Midterm Examination * Victor Matveev**

1. **(12pts)** Analyze the stability of equilibrium, and make a rough sketch of the solution, *y*(*t*). Hint: start by finding the first term in the Taylor series of the velocity field, but don't use direct differentiation!

$$\begin{cases} \dfrac{dy}{dt} = \ln\left(1-\sin^3 y\right)\cdot\left(\cos\left(\sin^2 y\right)-1\right)^2 \\ y(0) = 0.1 \end{cases}$$

2. **(20pts)** Sketch the flow field, and plot *x*(*t*) and *y*(*t*) for the given initial condition. Is it a Hamiltonian or potential flow? If so, find the Hamiltonian or the potential:
$$\begin{cases} \dfrac{dx}{dt} = -y; \quad x(0)=0 \\ \dfrac{dy}{dt} = -x^2; \quad y(0)=1 \end{cases}$$

3. **(20pts)** Consider the following reaction-advection PDE: $\dfrac{\partial\rho}{\partial t} + c\dfrac{\partial\rho}{\partial x} = -\gamma\dfrac{\rho^2}{K^2+\rho^2}$.

   Assume that the units are $[\rho] = M, \ [x] = L, \ [t] = T$

Determine the units of the constants $\gamma$, *c* and *K*. Non-dimensionalize this system (Hint: there is an obvious choices for the scale of $\rho$)

==================== **You may drop *one* of the remaining three problems** ====================

4. **(24pts)** Solve this PDE by the method of characteristics: $\begin{cases} \dfrac{\partial u}{\partial t} + 3xt^2\dfrac{\partial u}{\partial x} = 0 \quad (t>0; \ -\infty \le x \le +\infty) \\ u(x,0) = u_o(x) = \sin x \end{cases}$

   **a)** Find and plot the characteristics corresponding to 5 values of $x_0$: $x_0 = -2$, $x_0 = -1$, $x_0 = 0$, $x_0 = 1$, $x_0 = 2$.

   **b)** Find the solution, and check that it satisfies the given equation by direct differentiation

   **c)** Make a *rough* plot of the solution *u*(*x*,*t*) at *t*=1

5. **(24pts)** Make a rough plot of the solution (assume ε = const > 0), and find the 1st three terms in the asymptotic approximation to the solution for ε<<1: $\begin{cases} \dfrac{dy}{dt} = \exp(-\varepsilon y) \\ y(0) = 1 \end{cases}$

   Hint: substitute $y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \ldots$, and set $y_0(0) = 1, \ y_1(0) = y_2(0) = 0$

6. **(24pts)** Examine the following reaction: $B + C \underset{k^-}{\overset{k^+}{\rightleftharpoons}} A$

   a) Write down the system of differential equations describing reactant concentrations *A*, *B*, and *C*

   b) Use conservation laws to eliminate the concentrations of *B* and C from your equations, leaving only a single equation for the concentration of *A*, given the initial conditions $B_0 = C_0 = 1$, $A_0 = 0$

   c) Find the equilibrium value of *A*(t) and analyze its stability