Math 613 * Fall 2018 * Victor Matveev
Homework 1: units, nondimensionalization, and scaling

1. **(10pts)** Write down the quadratic Taylor polynomial for $f(x)=\ln(\cos(2x))$ near x=0, and use it to approximate *f*(0.2). Compare with a more accurate numerical result. Don't differentiate *f*(*x*): use Taylor series composition instead, recalling that $\ln(1+x)=x-x^2/2+x^3/3+O(x^4)$, $\cos(x)=1-x^2/2+O(x^4)$

2. **(20pts)** The bi-molecular binding reaction of recombination X+X→Y is described by the following differential equation for the number of molecules of X, or its volume concentration, *n*(*t*):

$$\begin{cases} \dfrac{dn}{dt} = -k\,n^2 \\ n(0) = n_0 \end{cases}$$

   a) Assuming that the physical units of *n* is volume density, $[n]=1/L^3$, and that *t* is time with units [*t*]=T, find the physical units of rate constant *k*. Then, non-dimensionalize this system.

   b) Now, suppose that *n* represents the number of molecules of X rather that its volume density. Explain **in one sentence** why d*n*/d*t* is proportional to n^2 rather than the 1st power of *n*. Hint: consider the change in molecule number of X over a small time step Δt, as we did in class for the degradation reaction. Assume that the particles are well-mixed within a given volume, and all particles have a chance to interact with each other, even in a small time step.

3. **(50pts)** Rocket blasts off from the Earth's surface. During the initial phase of flight, fuel is burned at the maximal possible rate α, and the exhaust gas is expelled downward with velocity β relative to the velocity of the rocket. The motion is governed by the following generalized Tsiolkovsky equation (note that $t<M_0/\alpha$, but that's not important for this assignment):

$$\begin{cases} \dfrac{dm}{dt} = -\alpha, \quad m(0) = M_0 = const \\ \dfrac{dx}{dt} = v(t), \quad x(0) = 0 \\ \dfrac{dv}{dt} = \dfrac{\alpha\beta}{m(t)} - \dfrac{g}{\left[1+x(t)/R\right]^2}, \quad v(0) = 0 \end{cases}$$

   The variables are:

$$\begin{cases} m(t) = \text{Mass of the rocket} \\ \text{v}(t) = \text{Upward velocity} \\ \text{x}(t) = \text{Height above Earth's surface} \end{cases}$$

   Parameters are (all are constant):

$$\begin{cases} \alpha = \text{Fuel burn rate (find its units)} \\ \beta = \text{Exhaust gas velocity relative to rocket} \\ M_0 = \text{Initial mass of the rocket} \\ g = \text{Acceleration of free fall near Earth's surface} \\ R = \text{Radius of the Earth} \end{cases}$$

   Non-dimensionallize this problem, but do not solve. Hint: use the most obvious scales for *m*(*t*), *x*(*t*) and *v*(*t*). Then, examine the first equation (dm/dt) to find the time scale. Write down the system in terms of non-dimensional variables and two non-dimensional parameters (call them *p* and *q*), which depend on the original dimensional parameters.

4. **(20pts)** Repeat the non-dimensionalization in problem 3, but now use another time scale, by non-dimensionalizing the second equation for dx/dt first, before non-dimensionalizing the equation for dm/dt.