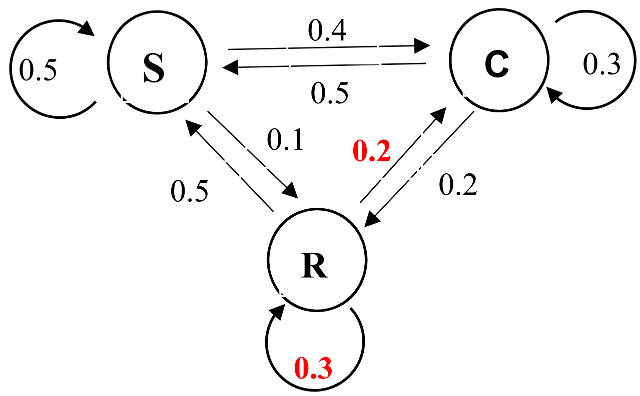
Math 613 * Fall 2019 * Victor Matveev * Homework #12

1. Consider the **discrete state, discrete time** Markov Chain describing a toy weather model, with daily transitions between "S" (sunny), "C" (cloudy) and "R" (rainy) days (which were supposedly obtained using repeated observation):



a) Write down the Markov Matrix. What is the equilibrium weather probability distribution?

b) Find *all* eigenvectors and eigenvalues. If you prefer, you can use MATLAB or Mathematca / Wolfram Alpha.

c) Write down the explicit solution of this discrete-time dynamical system, assuming that the weather was sunny on day zero. Hint: $\mathbf{p}^m \equiv \mathbf{p}(t_m) = M^m \mathbf{p}^0 = V\Lambda^m V^{-1}\mathbf{p}^0 = \sum_{k=1}^{3} \lambda_k^m c_k \mathbf{v}_k = \sum_{k=1}^{3} \lambda_k^m \mathbf{U}_k; \quad \mathbf{p}^0 = \sum_{k=1}^{3} \mathbf{U}_k$

d) What is the probability that it is cloudy on day 10, given that it is sunny on day zero?

2. Consider **the discrete state, continuous time** process describing the following chemical system (which is also a model for a biological population growth):

$$2A \xrightarrow{r_B} 3A$$
$$A \xrightarrow{r_D} \varnothing$$

Since the number of distinct pairs among n particles equal n (n–1) / 2, the Chemical Master Equation (CME) system is:

$$\begin{cases} \dfrac{dp_0}{dt} = r_D p_1 \\ \dfrac{dp_1}{dt} = r_D\left[2p_2 - p_1\right] \\ \dfrac{dp_n}{dt} = r_D\left[(n+1)p_{n+1} - np_n\right] + \dfrac{r_B}{2}\left[(n-2)(n-1)p_{n-1} - n(n-1)p_n\right] \quad (n>1) \end{cases}$$

a) To understand this system of equations, draw the Markov diagram and write down the first few rows of the transition matrix W describing this linear CME system, $\dfrac{d\mathbf{p}}{dt} = W\,\mathbf{p}$

b) Multiplying the master equation by n and summing over all n, obtain an ODE describing the evolution of the average number of particles $\langle n \rangle$: $\dfrac{d\langle n\rangle}{dt} = \sum_{n=1}^{\infty} n \dfrac{dp_n}{dt}$. Compare the resulting equation with the deterministic ("mass-action") ODE for the average number of particles: $\dfrac{dn}{dt} = \dfrac{r_B}{2} n(n-1) - r_D n$

c) Find the PDE for the probability generating function, $F(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n$.

Hint: $\dfrac{\partial F}{\partial z} = \sum_{n=0}^{\infty} n\, p_n(t) z^{n-1}, \quad \dfrac{\partial^2 F}{\partial z^2} = \sum_{n=0}^{\infty} n(n-1) p_n(t) z^{n-2}$.