Math 613 * Fall 2019 * Victor Matveev * Homework 4

1. **(10pts)** Find the first non-zero term in the Taylor series of function $f(x)=\dfrac{\ln\left(\cos\left(2x^4\right)\right)\sin^3\left(\sin^2 x\right)}{x^2\left(e^{2\tan x}-1\right)^3}$ near x=0, assuming that the removable singularity is removed. This should take less than 2 minutes if you use the composition of Taylor series. Make sure to check your answer by computing $f(0.01)$.

2. **(10pts)** Show that the following flow is both incompressible and irrotational (both the divergence and the curl of the velocity field are zero). Therefore, both the velocity potential and the Hamiltonian (the streamline function) exist. Find them both, and sketch the flow in the plane. Note that the flow is singular at the origin. Hint: these two functions form the real and imaginary parts of a well-known elementary complex function.

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{x^2+y^2} \\ \dfrac{dy}{dt} = \dfrac{y}{x^2+y^2} \end{cases}$$

3. **(40pts)** Examine the chemical reactor for *B* production:

$$\left\{ A+A \underset{k^-}{\overset{k^+}{\rightleftharpoons}} B; \quad \varnothing \xrightarrow{k_A} A; \quad B \xrightarrow{k_B} \varnothing \right\}$$

   a) Non-dimensionalize this system using time scale $t_c = 1/k^-$ and concentration scale $A_c = B_c = k^-/k^+$ (this ratio is called the "affinity" of a 2nd order (bimolecular) chemical reaction).

   b) Find the equilibrium, and analyze the linear stability of the equilibrium.

   c) For the special case $k_B = k^-$, sketch the nullclines in the phase plane, and show the trajectory for the initial condition *A*(0)=*B*(0)=0

4. **(40pts)** The SIR (susceptible-infected-recovered) epidemiology model with immunity extinction is a loop reaction chain

$$\begin{cases} S+I \xrightarrow{k_I} 2I \\ I \xrightarrow{k_R} R \xrightarrow{k_E} S \end{cases}$$ which is alternatively also denoted as $$S \xrightarrow{I\cdot k_I} I \xrightarrow{k_R} R \xrightarrow{k_E} S$$

   a) Using the conservation law $\left(S(t)+I(t)+R(t)=const=N\right)$, eliminate variable *R*(t)

   b) Non-dimensionalize the system of 2 ODEs, using time scale $t_c = \dfrac{1}{N k_I}$, and concentration scale $S_c = I_c = R_c = N$

   Denote $\rho_R = \dfrac{k_R}{N k_I}; \ \rho_E = \dfrac{k_E}{N k_I}$

   c) Sketch the nullclines and the flow field in the phase plane, assuming $\rho_R < 1$

   d) Find all equilibria and analyze their linear stability, assuming $\rho_R < 1$