Math 613 * Fall 2019 * Victor Matveev * Homework #6

Problem 1 (20pts) Derive the diffusion / heat equation in a one-dimensional tube/cable for the case where the tube cross-section varies along the length, $A(x)$. Then, consider the special case $A(x) = x^2$. Does the resulting equation remind you of anything from Calculus III?

Hint: Fick's law of diffusion $J = -D\frac{\partial u}{\partial x}$ remains unchanged; you only have to modify the conservation law derivation, and then combine the two equations. As a reminder, below is the derivation of the conservation law for the case of constant cross-section (modify it for the case where $A=A(x)$, not constant):

$$\frac{dN(t)}{dt} = \frac{\partial}{\partial t}\left(u\left(x^*,t\right)\Delta V\right) = \frac{\partial u\left(x^*,t\right)}{\partial t}A\Delta x = (\text{inflow rate}) - (\text{outflow rate}) = AJ(x,t) - AJ(x+\Delta x,t)$$

$$\Rightarrow \frac{\partial u\left(x^*,t\right)}{\partial t} = -\frac{J(x+\Delta x,t) - J(x,t)}{\Delta x} \quad \Rightarrow \quad \frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x}$$

Problem 2 (15pts) Find the truncation error for the following centered difference approximation of $u'(x)$:

$$\frac{du}{dx}(x_n) \approx \frac{u_{n+1} - u_{n-1}}{2\Delta x}$$

To do this, expand $u_{n+1} = u(x_{n+1}) = u(x_n + \Delta x)$ and $u_{n-1} = u(x_{n-1}) = u(x_n - \Delta x)$ in a Taylor series around point x_n, as we did in class when analyzing the centered difference approximation of the 2nd derivative. Finally, check the accuracy of this approximation by applying it to the function $u(x)=x^3$

Problem 3 (35pts)

Use MATLAB or Mathematica (or any other language) to find the solution to $\begin{cases} u_t = Du_{xx} - \gamma u \;\; (0 < x < 1) \\ u(0,t) = 0; \;\; u(1,t) = 10 \\ u(x,0) = 10x \end{cases}$ over **10 time steps** with $D = 0.2$, $\gamma = 10$, $\Delta t = 0.01$ and N=10 subdivisions along the x-interval [0, 1].

Hint: find u_n^{m+1} from the forward Euler discretization of this equation: $\frac{u_n^{m+1} - u_n^m}{\Delta t} = D\frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{(\Delta x)^2} - \gamma u_n^m$

Present the result as a table and plot these values as a surface using MATLAB / Mathematica / anything

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Problem 4 (30pts)

Find the ***equilibrium*** solution $u_{eq}(x)$ to the heat equation with heat loss along the length of the cable:

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \gamma u(x,t) \quad (0 < x < L,\; t > 0;\; \gamma = const > 0;\; D = const > 0) \\ \frac{\partial u}{\partial x}(0,t) = 0; \;\; u(L,t) = T = const \\ u(x,0) = 0 \end{cases}$$

Make a rough plot by hand of $u_{eq}(x)$ and explain the heat balance: where does the heat enter, and where does it exit at equilibrium? **See next page for a hint**

Hint: at equilibrium, time derivative is zero, therefore

$$\begin{cases} D\dfrac{d^2u_{eq}(x)}{dx^2} - \gamma u_{eq}(x) = 0 \quad (0 < x < L, \;\; \gamma = const) \\ \dfrac{du_{eq}}{dx}(0) = 0; \;\; u_{eq}(L) = T = const \end{cases}$$