Math 613 * Fall 2019 * Victor Matveev * Homework 9

Boldface quantities are constant vectors or vector fields; *italic* quantities are scalars: $|\mathbf{u}| \equiv u,\ \hat{\mathbf{u}} = \dfrac{\mathbf{u}}{|\mathbf{u}|}$

1. "Verify" the divergence theorem in $\mathbb{R}^2$ for the vector field $\mathbf{u}(x, y) = (xy^2, -x^2)$ and a square volume V: $V=[-1,+1] \times [-1,+1]$. Sketch this vector along the perimeter of the domain to check the signs of your integrals on the two sides of the divergence theorem.

2. Use divergence theorem to solve the Gauss equation with a point source in $\mathbb{R}^2$ (i.e. on a plane): $\nabla \cdot \mathbf{E}(\mathbf{r}) = q\delta(\mathbf{r})\ (q = const)$. Assume angle-independence. Finally, find the potential $\Phi(r)$ by solving $\mathbf{E} = -\nabla\Phi$, up to an integration constant (note: potential is not bounded in $\mathbb{R}^2$). Hint: Use a disk of fixed radius as your integration domain. In the case of angle symmetry, the gradient in polar coordinates becomes $\nabla\Phi = \dfrac{d\Phi}{dr}\hat{\mathbf{r}}$, therefore $E(r) = -\dfrac{d\Phi}{dr}$

3. Use **two** methods outlined below to find the electric field $\mathbf{E}(r)$ and electric potential $\Phi(r)$ both inside and outside a sphere or radius r_1 with charge density $\rho(\mathbf{r}) = \begin{cases} \rho_0 = const, & r_0 \le r \le r_1 \\ 0, & r < r_0 \text{ or } r > r_1 \end{cases}$. Make a rough plot of $E(r)$ and $\Phi(r)$ that you found. Assume that $\mathbf{E}(r)$ is oriented away from the center of the sphere, and that its magnitude depends only on the distance from the center: $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}},\ E \equiv |\mathbf{E}|,\ \hat{\mathbf{r}} = \dfrac{\mathbf{r}}{|\mathbf{r}|} \equiv \dfrac{\mathbf{r}}{r}$

a) Apply the divergence theorem to the Gauss law $\nabla \cdot \mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})/\varepsilon_0$, with spheres of constant radius R, considering three cases: $R < r_o$, $r_o < R < r_1$ and $R > r_1$. Then, calculate the electric potential, noting that $\mathbf{E}(\mathbf{r}) = -\nabla\Phi = -\dfrac{d\Phi}{dr}\hat{\mathbf{r}} \quad \Rightarrow \quad E(r) = -\dfrac{d\Phi}{dr}$. Set the integration constants so that $\Phi(r)$ and $\Phi'(r)$ are continuous functions.

b) Carry out the same solution by directly solving the Poisson equation $\nabla^2\Phi = -\rho(\mathbf{r})/\varepsilon_0$. Set the integration constants so that $\Phi(r)$ and $\Phi'(r)$ are continuous:

$$\text{Hint: } \begin{cases} \nabla^2\Phi = \dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{d\Phi}{dr}\right) = -\dfrac{\rho_0}{\varepsilon_0} \quad (r_0 < r < r_1) \\ \nabla^2\Phi = \dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{d\Phi}{dr}\right) = 0 \quad (r < r_0 \text{ or } r > r_1) \\ \Phi(r \to \infty) \to 0 \\ \Phi(r_0^-) = \Phi(r_0^+);\quad \Phi(r_1^-) = \Phi(r_1^+); \\ \Phi'(r_0^-) = \Phi'(r_0^+);\ \Phi'(r_1^-) = \Phi'(r_1^+) \end{cases}$$