Math 656 Complex Variables I

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Contents

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Book: M.J. Ablowitz & A.S. Fokas. Complex variables: Introduction and Applications (2nd edition). *Cambridge University Press* (2003).

1 Introduction to the course

You are familiar with the theory and calculus of functions of one (or more) real variable, $f(x)$, $f(x, y)$, $f(x, y, t)$ etc. This course is concerned with the theory of complex-valued functions of a complex variable:

$$f(z) = u(x, y) + iv(x, y), \quad \text{where} \quad z = x + iy \quad \text{and} \quad i^2 = -1.$$

We begin by introducing complex numbers and their algebraic properties, together with some useful geometrical notions.

1.1 Complex numbers

The set of all complex numbers is denoted by $\mathbb{C}$, and is in many ways analogous to the set of all ordered pairs of real numbers, $\mathbb{R}^2$. A complex number is specified by a pair of real numbers (x, y): we write $z = x+iy$, where $i^2 = -1$. We say that x is the **real part** of z, and y is the **imaginary part** of z, using the equivalent notation

$$x = \Re(z) = \mathrm{Re}(z), \quad y = \Im(z) = \mathrm{Im}(z).$$

Note that the relation $i^2 = -1$ leads to the identities

$$i^{2m} = (-1)^m, \quad i^{2m+1} = (-1)^m i, \quad \text{for } m \in \mathbb{Z}. \tag{1}$$

Two complex numbers are equal if and only if both their real and imaginary parts are equal. The **absolute value** or **magnitude** of the complex number z is defined by the length of the vector (x, y) associated with z:

$$|z| = \sqrt{x^2 + y^2}, \tag{2}$$

always a positive quantity (except when $z = 0$).

Example 1.1 *The complex number $z_1 = 3+4i$ has magnitude $|z| = \sqrt{3^2 + 4^2} = 5$, as has the complex number $z_2 = 4 + 3i$.*

The set of all real numbers $\mathbb{R}$ is a natural subset of $\mathbb{C}$, being the set of all those numbers in $\mathbb{C}$ with zero imaginary part. The standard geometry of $\mathbb{R}^2$ provides a convenient and useful representation of $\mathbb{C}$ (this geometrical structure is known as the **complex plane**: the real numbers lie along the x-axis, known as the real axis, and the pure imaginary numbers lie along the y-axis, known as the imaginary axis).

It is also often helpful to use the polar representation (r, θ) of points in 2D space, where $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ (alternatively, $x = r\cos\theta$ or $y = r\sin\theta$). In this representation,

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

Here again r is the absolute value of z, $r = |z|$, and the angle θ is known as the **argument** of z, written $\theta = \arg(z)$. Note that θ in this representation of z is not single-valued: if $z_0 = r_0(\cos\theta_0 + i\sin\theta_0)$ is one representation, then replacing θ_0 by $\theta_0 + 2n\pi$ also gives the same complex number z. Any such value of θ is an argument of the complex number, and $\arg(z)$ is therefore really an infinite **set** of θ-values. It is often useful to define a unique **principal argument**, which is usually taken to be the argument lying in the range $0 \leq \theta < 2\pi$ (but sometimes the range $-\pi < \theta \leq \pi$; convenience usually dictates the choice in a particular application). The principal argument is sometimes written $\theta = \mathrm{Arg}(z)$ to distinguish it from the set $\arg(z)$. We will see further consequences of this nonuniqueness later on, when we discuss multi-valued functions.

Example 1.2 *Find the polar form of the complex numbers (i) $z_1 = i$, (ii) $z_2 = 1 - i$, (iii) $z_3 = \sqrt{3} - i$.*

Solution: (i) For z_1 we have $|z_1| = 1$ and $\theta = \cos^{-1}(0) = \sin^{-1}(1) = \pi/2 + 2n\pi$, for any integer n. The principal argument would be $\theta = \pi/2$. In any case it is clear that $z_1 = \cos(\pi/2) + i\sin(\pi/2)$.
(ii) For $z_2 = 1 - i$ we have $|z_2| = \sqrt{2}$, and thus $1 = \sqrt{2}\cos\theta$, $-1 = \sqrt{2}\sin\theta$. Thus $\theta = -\pi/4$, modulo 2π, and $z_2 = \sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$.
(iii) $|z_3| = 2$, and $\sqrt{3} = 2\cos\theta$, $-1 = 2\sin\theta$. Thus $\theta = -\pi/6$ (modulo 2π), and $z_3 = 2(\cos(-\pi/6) + i\sin(-\pi/6))$.

1.1.1 Exponential representation of complex numbers

A more concise representation of z is obtained by introducing the complex exponential function:

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}, \tag{3}$$

where θ is the polar angle introduced above. This definition is a special case of the more general complex exponential function that we will introduce later. For now we note that the definition makes sense in terms of the standard Taylor series representation of these functions:

$$\begin{aligned} e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} &= \sum_{m=0}^{\infty} \frac{(i\theta)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(i\theta)^{2m+1}}{(2m+1)!} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m\theta^{2m}}{(2m)!} + i\sum_{m=0}^{\infty} \frac{(-1)^m\theta^{2m+1}}{(2m+1)!} \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

using the identities (1). The absolute value of the complex number $w = e^{i\theta}$ is clearly $|w| = 1$, and as the argument θ varies, the point representing $e^{i\theta}$ in the complex plane moves around the unit circle centered on the origin. In general then, we have the equivalent representations

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{4}$$

This last representation provides an easy way to take the power of a complex number:

$$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Note that this gives a simple derivation of the identity $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, a result known as De Moivre's theorem.

Example 1.3 *For the numbers in example 1.2 we have (i)* $z_1 = e^{i\pi/2}$*, (ii)* $z_2 = \sqrt{2}e^{-i\pi/4}$*, and (iii)* $z_3 = 2e^{-i\pi/6}$.

1.1.2 Algebra of complex numbers

Recalling the identities (1) it is straightforward to extend the algebra of the real numbers to complex numbers. If $z = x + iy$ and $w = u + iv$, then

$$
\begin{aligned}
z + w &= x + u + i(y + v), \\
zw = (x + iy)(u + iv) &= xu - yv + i(xv + yu).
\end{aligned}
$$

From these laws and the properties already noted, it is easy to show that the complex numbers $\mathbb{C}$ form a **field**. The inverse of the complex number z, given by $1/z$, satisfies

$$
z^{-1} = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2}. \tag{5}
$$

This identity may be alternatively obtained by the complex polar representation (4), which gives

$$
z^{-1} = \frac{e^{-i\theta}}{r} = \frac{re^{-i\theta}}{|z|^2}. \tag{6}
$$

The polar representation also gives the neatest interpretation of complex multiplication (and division which, with the notion of an inverse given above is really just multiplication also), since if $z = re^{i\theta}$ and $w = Re^{i\phi}$ then

$$
zw = rRe^{i(\theta+\phi)}.
$$

Example 1.4 *The product of* z_1 *and* z_2 *in example 1.2 is* $z_1 z_2 = \sqrt{2}e^{i\pi/4} = 1 + i$*; the product of* z_1 *and* z_3 *is* $z_1 z_3 = 2e^{i\pi/3} = 1 + i\sqrt{3}$*; and the product of* z_2 *and* z_3 *is* $z_2 z_3 = 2\sqrt{2}e^{-5i\pi/12}$.

1.1.3 Complex conjugates and inequalities

Every complex number $z = x + iy$ has a **complex conjugate** $\overline{z}$, defined by

$$
\overline{z} = x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}. \tag{7}
$$

The first equality above shows that geometrically, the complex conjugate of a point $z \in \mathbb{C}$ is obtained by **reflection** of z in the real (x) axis. The definition

(7) also shows that the inverse of z, given by (5) or (6) above, is related to its complex conjugate by

$$z^{-1} = \frac{\overline{z}}{|z|^2}.$$

It is evident that

$$\overline{\overline{z}} = z, \quad z + \overline{z} = 2\Re(z), \quad z - \overline{z} = 2i\Im(z), \quad |\overline{z}| = |z|, \quad z\overline{z} = |z|^2.$$

Moreover, for two complex numbers z and w,

$$\overline{z+w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z}\,\overline{w}.$$

The above properties show that $|zw| = |z||w|$, but in general $|z + w| \neq |z| + |w|$. Rather, the following inequalities hold, analogous to the **triangle inequalities** of vector algebra:

$$||z| - |w|| \leq |z + w| \leq |z| + |w|. \tag{8}$$

The second inequality may be proved using elementary properties of complex numbers. First note that

$$|z + w|^2 = (z + w)(\overline{z} + \overline{w}) = |z|^2 + |w|^2 + 2\Re(z\overline{w}),$$

and then that, for any complex number, the absolute value of its real part must be less than or equal to the absolute value of the complex number itself, which gives

$$\Re(z\overline{w}) \leq |\Re(z\overline{w})| \leq |z\overline{w}| = |z||w|.$$

Thus,

$$|z + w|^2 \leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2.$$

Noting that $|z+w| \geq 0$ and $|z|+|w| \geq 0$, the second inequality of (8) follows. For the first inequality, we use the result just proved, and set $Z = z + w$, $W = -w$ (arbitrary complex numbers, since z and w are arbitrary). Then $z = Z + W$, $w = -W$, and

$$|Z| \leq |Z + W| + |W| \quad \Rightarrow \quad |Z| - |W| \leq |Z + W|.$$

If $|Z| \geq |W|$ then the result follows; if not then repeat the above argument with $W = z + w$ and $Z = -w$ to obtain

$$|W| - |Z| \leq |Z + W|,$$

proving the result for $|W| \geq |Z|$ also.

The right-hand inequality in (8) generalizes easily to an arbitrary sum of complex numbers:

$$\left|\sum_{n=1}^{N} z_n\right| \leq \sum_{n=1}^{N} |z_n|.$$

When the complex numbers z, w are plotted in the complex plane $|z|$ is just the length of the straight line joining z to the origin. The complex number $z+w$ is found from the usual parallelogram law for vector addition. Therefore (8) has a geometrical interpretation in terms of lengths of sides of a triangle.

Exercise: Determine the conditions under which equality holds in (8).

Example 1.5 *For the complex numbers $z = 3 + 0i$ and $w = 0 + 4i$, $|z| = 3$, $|w| = 4$, and $|z + w| = |3 + 4i| = 5$. Clearly,*

$$||z| - |w|| = 1 \leq |z + w| = 5 \leq |z| + |w| \leq 7.$$

Example 1.6 *For the complex numbers $z = 4$, $w = 3$, $|z|+|w| = 7 = |z+w|$. Similarly for the complex numbers $z = 4i$, $w = 3i$.*

Example 1.7 *For $z = 4$, $w = -3$, $||z| - |w|| = 1 = |z + w|$. Similarly for $z = 4i$, $w = -3i$.*

Homework: 1. Prove that, for $0 \neq z \in \mathbb{C}$, $|z| \leq |\Re(z)| + |\Im(z)| \leq \sqrt{2}|z|$. Show, by examples, that either (but not both!) of these inequalities may be an equality.

2. Prove that, for $z, w \in \mathbb{C}$,

$$|1 - \overline{z}w|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2).$$

Deduce that, if $|z| < 1$ and $|w| < 1$,

$$\left|\frac{z - w}{1 - \overline{z}w}\right| < 1.$$

Remark Note that inequalities pertain only to the **absolute value** of complex numbers. Unlike the real numbers, there is **no ordering** of the field of complex numbers, so no notion of one complex number being greater than or less than another.

1.2 Solving simple complex equations

Algebraic complex equations may be solved numerically in the same way as real equations in two variables: by varying $x = \Re(z)$ and $y = \Im(z)$ until a zero is found. In sufficiently simple cases analytical solutions may be written down. Consider the general quadratic equation $az^2 + bz + c = 0$, for a, b, $c \in \mathbb{C}$. The algebraic rules for complex numbers enable us (at least in principle!) to complete the square and solve "as usual":

$$\begin{aligned} & az^2 + bz + c = 0 \quad a, b, c \in \mathbb{C} \\ \Rightarrow \quad & a\left(z + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0 \\ \Rightarrow \quad & z = -\frac{b}{2a} + \left(\frac{b^2}{4a^2} - \frac{c}{a}\right)^{1/2}. \end{aligned}$$

We will see later how to take the square-root of the complex number $b^2/(4a^2) - c/a$ and obtain two distinct roots, just as in the real case. [**Note:** for quadratic equations with complex coefficients, the roots do **not** occur in complex conjugate pairs.]

Higher order polynomials are more difficult to solve analytically, but the equation $z^n = a$, for $a \in \mathbb{C}$, has a general solution, which we find by writing a using the complex polar representation (4) and using the multi-valued nature of the argument:

$$z^n = a = |a|e^{i(\phi + 2m\pi)} \quad m \in \mathbb{Z}. \tag{9}$$

Obviously, for any $m \in \mathbb{Z}$ the right-hand side of (9) is the same, but when we take the nth root to solve for z, different m-values give different roots for z. Thus

$$z_m = |a|^{1/n} e^{i(\phi + 2m\pi)/n} \quad m = 0, 1, \ldots n - 1, \tag{10}$$

gives n distinct solutions; and for $m \geq n$ this set of solutions is repeated ($m = n$ gives the same result as $m = 0$, $m = n + 1$ the same result as $m = 1$, and so on). In the special case $a = 1 = 1.e^{i.0}$ we have

$$z_m = e^{2m\pi i/n} = \omega_n^m, \quad m = 0, 1, \dots n-1;$$

these are the n roots of unity ($\omega_n = e^{2\pi i/n}$). More generally, any nth order polynomial equation $\sum_{j=1}^{n} a_j z^j = 0$ has exactly n complex solutions – we will prove this result later.

Example 1.8 *If $n = 2$ and $a = 1$ in (9), a familiar real case, (10) gives two solutions $z_m = e^{2m\pi i/2}$, $m = 0, 1$, that is, $z_0 = 1$ and $z_1 = e^{i\pi} = -1$.*
If $n = 3$ and $a = 1$ then we have the three cube roots of unity, $z_0 = 1$, $z_1 = e^{2\pi i/3}$, $z_2 = e^{4\pi i/3}$. Note that z_2 can be written equivalently as $z_2 = e^{-2\pi i/3}$.

Returning to the equation $z^n = 1$, we see that the factor $z - 1$ is easily isolated:

$$(z-1)(z^{n-1} + z^{n-2} + \dots + z + 1) = 0.$$

Clearly the solution $z = 1 = \omega_n^0$ is the solution corresponding to the first factor, and so each of $\omega_n, \omega_n^2, \dots, \omega_n^{n-1}$ satisfy the $(n-1)$th-order polynomial given by setting the second factor to zero. In fact, since we have all solutions to the equation, we know that we must be able to make the complete factorization as

$$z^n - 1 = (z-1)(z-\omega_n)\dots(z-\omega_n^{n-1}) = 0,$$

though it is far from obvious that the product of these n factors gives $z^n - 1$. Try showing that this is the case for small values of n, *e.g.*, $n = 3$, $n = 4$.

2 Subsets of the complex plane

2.1 Introduction: Simple subsets of the complex plane

Before saying more about complex valued functions of a complex variable, we need to introduce a few basic and important ideas about subsets of the complex plane, including the equations (or inequalities) that define such subsets, and geometrical interpretations. Many of these ideas will overlap with familiar concepts from the geometry of $\mathbb{R}^2$, since any geometrical object in the plane $\mathbb{R}^2$ has a direct analog in $\mathbb{C}$. We will also introduce the idea of parametrizing subsets of $\mathbb{C}$, which will be useful later on when we consider complex integration.

Perhaps the most basic subsets of $\mathbb{C}$ are lines, or line segments. For example, the real axis (the set of complex numbers with zero imaginary parts) may be described by any of the following equations:

$$\Im(z) = 0; \quad z = \overline{z}; \quad |z - i| = |z + i|.$$

More generally, noting that for any complex number $a \in \mathbb{C}$, $\overline{a}$ is its reflection in the real axis, we can state that the real axis is the set of all points that are equidistant from a and $\overline{a}$. This geometrical observation leads to the description of the real axis by the equation

$$|z - a| = |z - \overline{a}|, \quad \text{for any } a \in \mathbb{C} \text{ with } \Im(a) \neq 0.$$

Any straight line L in the complex plane can be described by a complex equation. This is easily obtained by noting that the line in $\mathbb{C}$ corresponding to the line $y = mx + c$ in $\mathbb{R}^2$ is found by substituting for $y = (z - \overline{z})/(2i)$, $x = (z + \overline{z})/2$, giving

$$z(1 - im) = \overline{z}(1 + im) + c.$$

As with $\mathbb{R}^2$, the half-spaces on either side of the line can be described by inequalities. Thus, the upper half-plane $\Im(z) \geq 0$ can be alternatively described by $|z - i| \leq |z + i|$, encoding the fact that points in the upper half z-plane lie closer to i than to $-i$; or alternatively, for $a \in \mathbb{C}$ with $\Im(a) \geq 0$, points in the upper half z-plane lie closer to a than to $\overline{a}$, thus $|z - a| \leq |z - \overline{a}|$.

Another way to express a line in $\mathbb{C}$ is by parametrization. The real axis $L_{\mathbb{R}}$ may be trivially parametrized as

$$L_{\mathbb{R}} = \{z \in \mathbb{C} : z = x,\ x \in \mathbb{R}\}.$$

If we wish to restrict to one or more line segments then we must restrict the value of the parameter x appropriately. In general, a line L of finite length between two points a, $b \in \mathbb{C}$ is defined by

$$L = \{z \in \mathbb{C} : z = (1 - t)a + tb, 0 \leq t \leq 1\}$$

though of course many other parametrizations are possible. Such a line segment is often also denoted by $L = [a, b]$.

Exercise: What subset is described by the parametrization

$$L = \{z \in \mathbb{C} : z = (1-t^2)a + t^2 b, 0 \leq t \leq 1\}?$$

What about

$$L = \{z \in \mathbb{C} : z = (1-t)a + t^2 b, 0 \leq t \leq 1\}?$$

What if the parameter t is unrestricted?

Circles have a very natural description in $\mathbb{C}$, as is evident from the polar description of complex numbers, (4). If r is fixed in (4) but θ varies from 0 to 2π then z moves around a circle of radius r in the complex plane. Since r is simply the modulus of the complex number, this circle may be written as $|z| = r$; the polar representation (4) provides the natural parametrization of the circle, with θ as parameter. More generally, a circle C centered on an arbitrary point $a \in \mathbb{C}$ has equation $|z - a| = r$, with parametric description

$$C = \{z \in \mathbb{C} : z = a + re^{i\theta},\ 0 \leq \theta < 2\pi\}.$$

The subset S of $\mathbb{C}$ lying within this circle is described by

$$S = \{z \in \mathbb{C} : |z - a| \leq r\}.$$

Annular regions between concentric circles may be described in the obvious way.

Another way to represent a circle in $\mathbb{C}$ is via the equation

$$\left|\frac{z-\alpha}{z-\beta}\right| = \lambda, \tag{11}$$

for α, $\beta \in \mathbb{C}$ ($\alpha \neq \beta$) and $0 < \lambda \in \mathbb{R}$, $\lambda \neq 1$. [The simplest way to show this is to switch to the cartesian representation $z = x + iy$ and square both sides in the above equation.] Note that this representation means that, if we define the new complex variable w by $w = (z-\alpha)/(z-\beta)$, then the **image** of the circle represented by (11) in the w-plane is another circle, $|w| = \lambda$. This is a simple example of a complex **mapping** (a Mobius transformation, in fact; complex mappings and their applications will be considered in detail in Math 756).

We can also describe **circular arcs** between two points in the complex plane. Again, various representations are possible, but the simplest makes

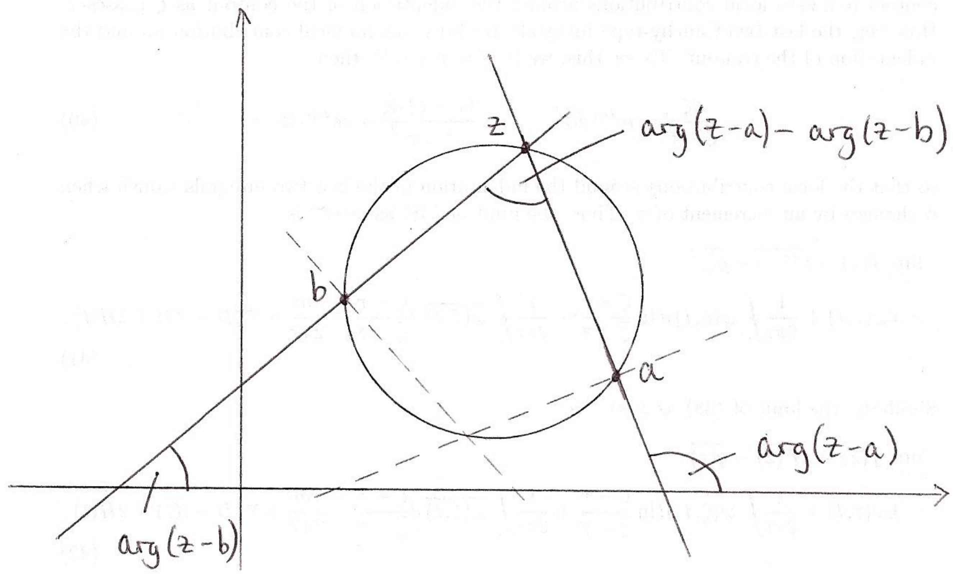

Figure 1: Two possible circular arcs passing through given points a, $b \in \mathbb{C}$

use of circle theorems of basic geometry. Suppose γ is a circular arc through points a, b, and z is an arbitrary point on γ. Then circle theorems tell us that the angle azb is a constant, say μ. We have

$$\arg(z-a) - \arg(z-b) = \mu$$

(see figure 1) or, using the fact that for any two complex numbers z_1 and z_2, $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$,

$$\arg\left(\frac{z-a}{z-b}\right) = \mu \mod 2\pi.$$

The case $\mu = \pi$ is the degenerate case when the arc collapses to a line segment joining a and b.

To find the equation of the other arc joining a and b that completes the circle, note that for a point w on that arc the angle awb must equal the supplement of azb, so that

$$\arg(b-w) - \arg(a-w) = \arg\left(\frac{b-w}{a-w}\right) = \arg\left(\frac{w-b}{w-a}\right) = \pi - \mu \mod 2\pi.$$

We now consider more general subsets of the complex plane, introducing several formal definitions.

2.2 Elementary definitions

Definition 2.1 *A* **neighborhood** N *of a point* $z_0 \in \mathbb{C}$ *is the set of points* z *such that* $|z - z_0| < \epsilon$.

This is sometimes referred to as an ϵ-neighborhood of z_0. Geometrically, in the complex plane, the neighborhood is the interior of a small disk of radius $\epsilon > 0$ centered on z_0, thus N is the set

$$N = \{z \in \mathbb{C} : z = z_0 + \delta e^{i\theta},\, 0 \le \delta < \epsilon,\, 0 \le \theta < 2\pi\}. \tag{12}$$

Since the inequality in the definition is strict, the boundary of the disk is excluded from the neighborhood (this point seems trivial, but is not, as we shall see). Related to neighborhoods is the concept of a **ball** or **disk** surrounding a given point z_0.

Definition 2.2 *The* **ball***, or* **disk***, of radius* r *about the point* z_0 *is denoted by* $B(z_0; r)$ *(or sometimes* $D(z_0; r)$*), and is defined by*

$$B(z_0; r) = \{z \in \mathbb{C} : |z - z_0| < r\}. \tag{13}$$

If we remove the point z_0 from the ball (or disk) the resulting set is called the **punctured** ball or disk, and is denoted by $B'(z_0; r)$ (or $D'(z_0; r)$).

Definition 2.3 *(Interior point) Let* $S \subset \mathbb{C}$ *be a subset of the complex numbers.* $z_0 \in S$ *is an* **interior point** *of the set* S *if* S *contains a neighborhood of* z_0.

Example: $z_0 = 1/2$ is an interior point of the set $S_1 = \{z \in \mathbb{C} : |z| < 1\} \subset \mathbb{C}$, but not of the set $S_2 = \{z \in \mathbb{C} : -1 < \Re(z) < 1, \Im(z) = 0\} \subset \mathbb{C}$.

Definition 2.4 *(Open set) The set* S *is* **open** *if all its points are interior.*

S_1 in the example above is open, but S_2 is not. To see that S_1 is open, we need to show that for any point $z_0 \in S_1$, we can find an ϵ-neighborhood of z_0 that is contained within S_1. Let $z_0 \in S_1$, then, by definition of S_1, $|z_0| < 1$.

Let $\epsilon = 1 - |z_0| > 0$. The ϵ-neighborhood $N_{z_0,\epsilon}$ of z_0 lies within S_1 because if $z_1 \in N_{z_0,\epsilon}$ then, by (12), z_1 can be written

$$\begin{aligned} & z_1 = z_0 + \delta e^{i\theta}, \quad \delta \in [0,\epsilon),\, \theta \in [0, 2\pi) \\ \Rightarrow \quad & |z_1| = |z_0 + \delta e^{i\theta}| \leq |z_0| + \delta = 1 - \epsilon + \delta < 1, \end{aligned}$$

where we used (8) in the above. Since $|z_1| < 1$, $z_1 \in S_1$, and since $z_1 \in N_{z_0,\epsilon}$ is arbitrary, we have shown that a neighborhood of z_0 lies within S_1 for any $z_0 \in S_1$. Hence all points of S_1 are interior points, and so S_1 is an open set.

To see that S_2 is not open we just need to find a point $z_0 \in S_2$ for which no neighborhood lies within S_2. In fact any point $z_0 \in S_2$ will do for this purpose. Using the description (12), a neighborhood of any $z_0 \in S_2$ contains all the points

$$z = z_0 + \delta e^{i\theta},\ 0 \leq \delta < \epsilon,\ 0 \leq \theta < 2\pi.$$

In particular, the point $z = z_0 + i\epsilon/2$ lies in any neighborhood of z_0 ($\delta = \epsilon/2$, $\theta = \pi/2$). Since $z_0 \in \mathbb{R}$ it follows that $\Im(z) = \epsilon/2 \neq 0$, and thus $z \notin S_2$. So no neighborhood of z_0 lies within S_2; z_0 is not an interior point, and S_2 is not open.

Definition 2.5 *(Closed set) A set S is* **closed** *if its complement $\mathbb{C} \setminus S = \{z \in \mathbb{C} : z \notin S\}$ is open.*

Definition 2.6 *(Limit point) A point $z_0 \in \mathbb{C}$ is a* **limit point** *of the set $S \subset \mathbb{C}$ if $B'(z_0; r) \cap S \neq \emptyset \ \forall \ r > 0$.*

Here the ball is punctured so that points of the set itself are not automatically limit points. A limit point must have other members of the set S clustering about it.

Example: If $S = \{(-1)^n i(1 + 1/n) : n = 1, 2, \ldots\}$ then S has $\pm i$ as limit points. It is neither open nor closed. The limit points do not lie in S.

Definition 2.7 *(Closure) The* **closure** *$\overline{S}$ of $S \subset \mathbb{C}$ is the union of S and its limit points.*

Very often, this will simply be the union of the set S and any of its boundary points that are not already included within S. As the name suggests, the closure of S is always a closed set, as the following lemma shows.

Lemma 2.8 *A set S is closed if and only if it contains all its limit points.*

Proof

$$\begin{aligned} S \quad \text{is closed} \quad &\iff \mathbb{C} \setminus S \,\text{is open} \\ &\iff \text{given}\, z \notin S,\, \exists\, \epsilon > 0 \,\text{such that}\, B(z, \epsilon) \subset \mathbb{C} \setminus S \\ &\iff \text{given}\, z \notin S,\, \exists\, \epsilon > 0 \,\text{such that}\, B'(z, \epsilon) \cap S = \emptyset \\ &\iff \text{no point of } \mathbb{C} \setminus S \text{ is a limit point of } S. \end{aligned}$$

Since we know the final statement is true if and only if the set contains all its limit points, the equivalence of this with the set being closed is proved.

Definition 2.9 *(Region) A* **region** *R is an open subset of $\mathbb{C}$, plus some, all or none of the boundary points.*

(**Note:** You will see different definitions of **regions** and **domains** (see later) in the complex plane – not all books are consistent!)

Example: S_1 (the open unit disc) is a region, as is $R_1 = S_1 \cup \{z \in \mathbb{C} : |z| = 1, 0 < \arg z < \pi/2\}$, and $R_2 = S_1 \cup \{z \in \mathbb{C} : |z| = 1\}$ (R_2 is the closed unit disc – the open unit disc plus all of its boundary points).

Example: R_2 in the example above is a closed region, but R_1 and S_1 are non-closed regions (S_1 is an open region as we showed, but R_1 is neither open nor closed, as it contains only some of its boundary points).

Definition 2.10 *A region R is said to be* **bounded** *if $\exists M > 0$ such that $|z| \leq M \, \forall z \in R$.*

Example: S_1, R_1 and R_2 above are all bounded by $M = 1$, but $R_3 = \{z \in \mathbb{C} : |z| > 1\}$ is unbounded. (A region that is both closed and bounded is **compact.**)

Homework: 1. Prove that any punctured ball $B'(z_0, r)$ is an open set. What are its limit points? What is its closure?

2.3 Stereographic projection

A one-to-one correspondence may be made between the "flat" complex plane and the surface Σ of a sphere of unit diameter in $\mathbb{R}^3$ (which we take as (x, y, Z)-space for the purposes of this discussion). Take the sphere to rest with its base (its South pole, $(0, 0, 0)$) at the origin of $\mathbb{C}$, $z = 0$. Then, for any $z = x+iy \in \mathbb{C}$, draw a straight line joining z to the North pole, $N = (0, 0, 1)$, of the sphere. This line intersects the surface of the sphere at a unique point $P(x, y)$, which depends only on the coordinates (x, y) associated with z. As the magnitude of the complex number $|z| \to \infty$, the point P approaches the North pole $(0, 0, 1)$, so that if we consider "∞" to be a point in $\mathbb{C}$ then this maps uniquely to the North pole of the sphere. This representation is sometimes referred to as a **compactification** of the complex plane; and the complex plane with this point "∞" appended is called the extended complex plane. A detailed description of such a correspondence may be found in Ablowitz & Fokas [2].

The explicit correspondence may be worked out, using elementary geometry, as

$$z = x + iy = re^{i\theta} \in \mathbb{C} \quad \leftrightarrow \quad \left(\frac{x}{1+r^2}, \frac{y}{1+r^2}, \frac{r^2}{1+r^2} \right) \in \Sigma$$

(the calculation is summarized in figure 2). This construction preserves many important geometrical properties of the complex plane. It is not hard to show that any circle on Σ (given by the intersection of a plane with Σ) projects to a circle in $\mathbb{C}$, except for circles that pass through the north pole N of Σ, which project onto **straight lines** in $\mathbb{C}$. Thus, straight lines in $\mathbb{C}$ may be thought of as "circles through infinity".

Homework: Ablowitz & Fokas, problems for section 1.2: Q 1.
If you read the extra section §1.2.2 in Ablowitz & Fokas on stereographic projection (optional!) you could also try Questions 10, 11, 12 from problems for section 1.2.

2.4 Paths in the complex plane

To develop concepts such as complex integration (which is always taken along a path or curve in $\mathbb{C}$), we need to think about how to describe such paths

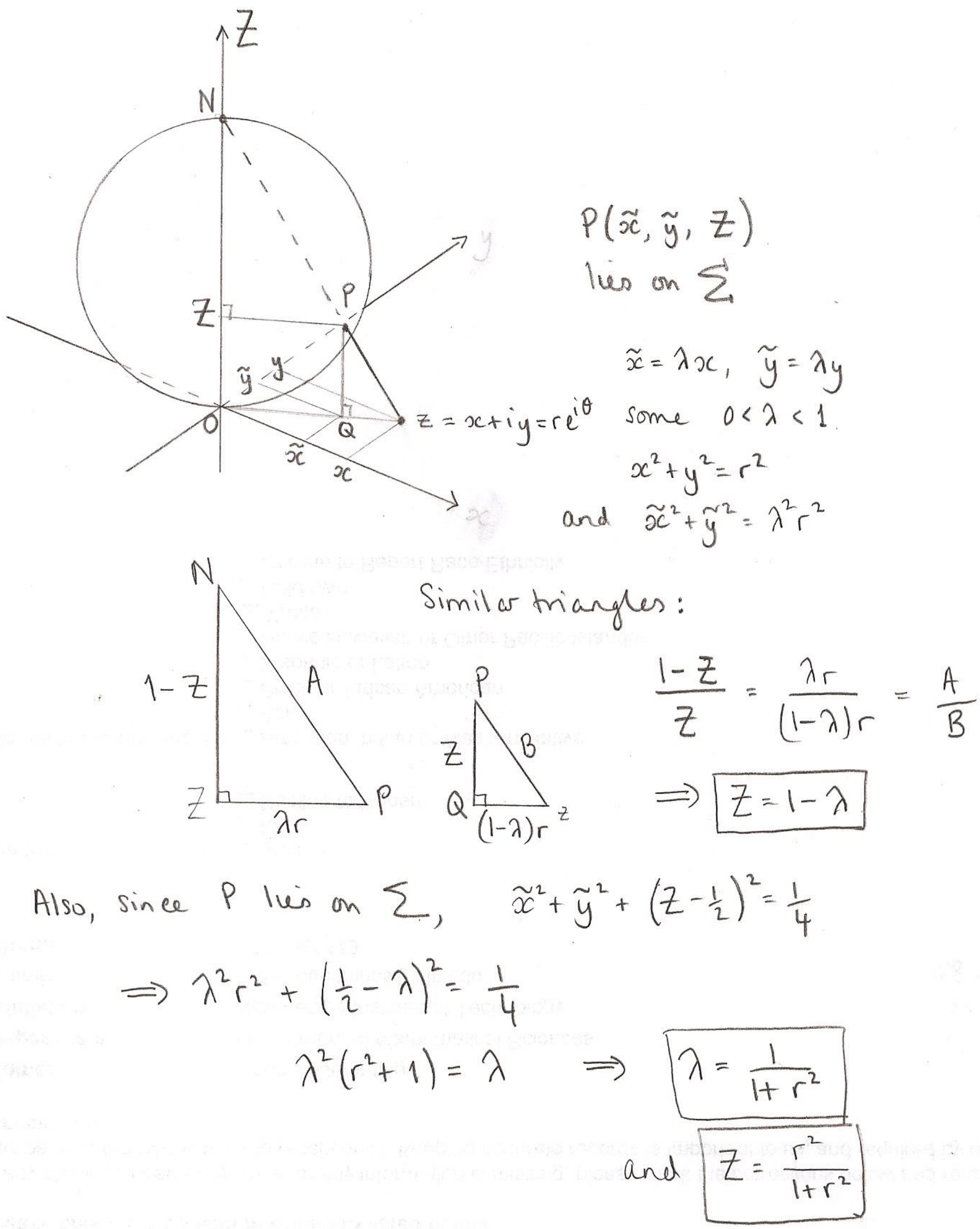

Figure 2: The stereographic projection of the extended complex plane onto the unit sphere.

through the complex plane. Paths are often defined in terms of a real parameter. We have already seen a couple of examples of simple paths that can be described in this way in §2.1. The circle, centered on z_0 with radius r, can be thought of as a path, whose end-point coincides with its beginning point (an example of a closed path). Paths are denoted variously by γ, Γ, or C, possibly with parametric dependence denoted too, *e.g.* $\gamma(t)$. If the path in question is the boundary of some domain D then the alternative notation ∂D may also be used. Thus, we might describe a circular path by

$$\gamma = \{z = z_0 + re^{i\theta} : 0 \leq \theta \leq 2\pi\}. \tag{14}$$

Here θ is the real parameter, and as θ increases from 0 to 2π we traverse the circle in the anticlockwise direction, starting at the point $z = z_0 + r$. The quantity $\gamma(\theta_0)$ denotes the point on the path corresponding to parameter value θ_0.

We also saw in §2.1 how a straight line between two points z_1 and z_2 may be represented parametrically, as

$$\gamma_1 = \{z = (1-t)z_1 + tz_2 : t \in [0,1]\} = [z_1, z_2]. \tag{15}$$

A path along a second line segment from z_2 to z_3 may obviously be defined in the same way:

$$\gamma_2 = \{z = (1-t)z_2 + tz_3 : t \in [0,1]\} = [z_2, z_3].$$

Continuing in this way, any sequence of points $z_n \in \mathbb{C}$ may be connected by a sequence of such line segments, forming a piecewise linear or **polygonal** path within $\mathbb{C}$.

A general path γ through complex space defined in terms of a real parameter t may be described by $z(t) = x(t) + iy(t)$, with t lying in some interval $[a, b]$. As for the specific examples above we write

$$\gamma = \{z(t) : t \in [a,b]\}. \tag{16}$$

Definition 2.11 *(Closed curve) A curve γ defined as in (16) is* **closed** *if $z(a) = z(b)$.*

Definition 2.12 *(Simple curve) A curve γ defined as in (16) is* **simple** *if it is non-self intersecting, that is, if $z(t_1) \neq z(t_2)$ for $a \leq t_1 < t_2 < b$ (the 2nd strict inequality here is to allow a closed curve to be simple).*

Examples (14) and (15) above are both simple paths. Example (14) is both simple and closed.

Homework: 1. Sketch the contours γ defined by: (i) $\gamma = \{z : z = 1 + ie^{it}, t \in [0, \pi]\}$. (ii) $\gamma = [-1, 1] \cup [1, 1 + i] \cup [1 + i, -1 - i] \cup [-1 - i, -1]$ (the join of these line segments). (iii) $\gamma = \{z : z = \cos te^{it}, t \in [0, 2\pi]\}$.
2. Define parametrically the following paths: (i) the square with vertices at $\pm 1 \pm i$; (ii) the closed semicircle in the right half-plane with $[-iR, iR]$ as diameter.

2.4.1 Paths and connectedness

Paths may be used to define a kind of **connectedness**, which is another concept that carries over from its analog in $\mathbb{R}^2$.

Definition 2.13 *(Connectedness) A subset or region S in the complex plane is said to be* **connected** *if it* **cannot** *be expressed as $S = S_1 \cup S_2$, where S_1 and S_2 are both non-empty, open, and* **disjoint.**

Definition 2.14 *(Polygonal path connectedness) S is said to be* **polygonally connected** *if, given any two points a, $b \in S$, there exists a polygonal path lying in S and having endpoints a and b.*

More generally, we have the concept of general **path-connectedness**, the idea that any two points within a set can be connected by a continuous path lying wholly within the set.

It is clear that each of these concepts is related. The following theorem makes this precise.

Theorem 2.15 *(Connectedness) A non-empty open set $S \subset \mathbb{C}$ is connected iff it is polygonally-connected.*

Proof Suppose S is connected. Let $a \in S$, and define S_a to be the set of points $z \in S$ that can be connected to a by a polygonal path lying within S. Let $S_0 = S \setminus S_a$.

Claim: Each of S_a and S_0 is open.

The connectedness of S will then imply that one of these two sets is empty. This cannot be S_0, since $a \in S_a$. Thus $S_a = S$, and S is also polygonally-connected.

Proof of claim: Let $z \in S$ be arbitrary. Since S is open and $z \in S$, $\exists\ \epsilon > 0$ such that $B(z;\epsilon) \subset S$. For each $w \in B(z;\epsilon)$ the line segment $[z,w] \subset B(z;\epsilon) \subset S$. It follows that z can be joined to a by a polygonal path in S **if and only if** w can be. Since $S = S_a \cup S_0$ and S_a, S_0 are disjoint, it follows that, for $k = a, 0$, $z \in S_k \Rightarrow B(z;\epsilon) \subset S_k$, and thus both S_a and S_0 are open, as claimed.

This proves the theorem one way, that a connected set is polygonally-connected. To prove the converse, suppose that S is non-empty, open, and polygonally-connected. We will use contradiction, and hence we suppose that $S = S_1 \cup S_2$, where both S_1 and S_2 are open, non-empty, and disjoint. Let $a \in S_1$ and $b \in S_2$, then there exists a polygonal path γ lying in S and joining a to b. At least one line segment of γ has one endpoint p in S_1, and the other, q, in S_2, and of course the entirety of γ is contained in $S_1 \cup S_2$. Set

$$\tilde{\gamma}(t) = (1-t)p + tq, \quad t \in [0,1],$$

and define the real function h of the real variable t by

$$h(t) = \begin{cases} 0 & \text{if} \quad \tilde{\gamma}(t) \in S_1, \\ 1 & \text{if} \quad \tilde{\gamma}(t) \in S_2. \end{cases}$$

Clearly $h(0) = 0$ and $h(1) = 1$, and h takes no intermediate values. If we can show that, nonetheless, h is continuous; then the required contradiction follows immediately.

To show that h is indeed a real-valued continuous function, we take a parameter value $s \in [0,1]$. The image of s, $z = \tilde{\gamma}(s)$, lies in one of S_1 or S_2; without loss of generality suppose it lies in S_1, so that $h(s) = 0$. S_1 is open, and so $\exists\ \epsilon_1 > 0$ such that $B(z,\epsilon_1) \subset S_1$.

Now let $t \in [0,1]$ with $0 < |s-t| < \epsilon_1/|p-q|$. Writing $w = \tilde{\gamma}(t)$ for the image of t, we now note that

$$|w - z| = |(1-s)p + sq - (1-t)p - tq| = |s-t||p-q| < \epsilon_1.$$

It follows that $w \in B(z,\epsilon_1) \subset S_1$, and hence that $h(w) = 0$. Hence, by choosing t sufficiently close to s we can always make $|h(t) - h(s)|$ as small as we like (zero, in fact). This means that h is a continuous function as claimed, contradicting the observed jump discontinuity. The set S must therefore be connected, if it is polygonally-connected.

Remark Implicit in the proof above was the notion of **convexity**, when we stated that for each $w \in B(z;\epsilon)$ the line segment $[z,w] \subset B(z;\epsilon)$. More generally we have the definition:

Definition 2.16 *(Convexity) Let $S \subset \mathbb{C}$. We say that S is* **convex** *if, given any pair of points a, $b \in S$, we have $[a,b] \subset S$.*

It is intuitively clear that any open ball is a convex set. A less stringent notion that is sometimes useful is that of a **starlike** subset:

Definition 2.17 *Let $S \subset \mathbb{C}$. We say S is starlike if $\exists$ $a \in S$ such that $[a,z] \subset S$ for all $z \in S$.*

All convex sets are starlike (compare the definitions), but not all starlike sets are convex. For example, a true star-shape is starlike (as its name suggests!) but not convex.
Example: All of the regions S_1, R_1, R_2, R_3 given in §2.2 previously are connected. The region $R_4 = S_1 \cup R_3$ is not connected, since no point in S_1 can be connected to a point in R_3 by a piecewise linear curve lying entirely within R_4. S_1, R_1, R_2 are convex, and therefore also starlike. R_3 and R_4 are neither convex nor starlike.

Definition 2.18 *A* **domain** *is a connected open region.*

Definition 2.19 *(Simply-connected) A domain D is said to be* **simply-connected** *if it is path-connected, and any path joining 2 points in D can be continuously transformed into any other.*

Example: All of the regions S_1, R_1, R_2 given above are simply-connected. The region R_3 is not, since any two distinct points can be joined by topologically-distinct paths within R_3, passing on either side of the "hole".

We conclude this section with one more important concept that will be needed later. We phrase the concept as a definition, though in fact it is really a theorem that can be proved.

Definition 2.20 *(Interior and exterior of a closed contour) Let γ be a simple closed contour. The complement of γ, $\mathbb{C} \setminus \{\gamma\}$, is of the form $I(\gamma) \cup E(\gamma)$, where $I(\gamma)$ and $E(\gamma)$ are disjoint connected open sets, and $I(\gamma)$ (the* **interior** *of γ) is bounded, while $E(\gamma)$ (the* **exterior** *of γ) is unbounded.*

This definition allows the **orientation** of a contour to be specified. We follow the standard convention in complex (and real) analysis, which is the following:

Definition 2.21 *(Orientation of a contour) A simple closed contour γ is said to be* **positively oriented** *if it is followed in the anticlockwise sense relative to any point in its interior.*

For a contour defined parametrically by (16), γ is positively oriented if the curve is followed counter-clockwise as the parameter t increases. Another way to remember is that, if you follow the curve around, its interior should stay on your left if it is positively oriented. Note that for a non-simple closed curve, different portions can have different orientations!

3 Analytic functions: The basics

3.1 Complex sequences, series, and functions

You are familiar with the properties and convergence/continuity criteria for real-valued sequences, series and functions. These concepts are very similar in the complex plane, and carry over quite naturally from the real case. Thus, for sequences (the simplest case to consider), we have the following:

Definition 3.1 *A complex sequence (z_n) is an assignment of a complex number z_n to each $n \in \mathbb{N}$.*

The notion of a subsequence may be defined exactly as for real sequences (an infinite, ordered, selection of elements of the original sequence).

Definition 3.2 *(Convergence of a sequence) The sequence (z_n)* **converges** *to a limit $a \in \mathbb{C}$ ($z_n \to a$) as $n \to \infty$ if, given $\epsilon > 0$, $\exists N_\epsilon$ such that $\forall\, n \geq N_\epsilon$, $|z_n - a| < \epsilon$.*

As in the real case, a series is simply the sum of the associated sequence; thus, the series associated with (a_n) is just $\sum_n a_n$, where the sum is taken over all elements of the sequence.

Definition 3.3 *(Convergence of a series) The series $\sum a_n$ is said to converge to the sum S if the sequence (S_n) of partial sums, $S_n = \sum_{m=1}^{n} a_m$, converges to the limit S in the sense of definition 3.2.*

Many of the results and theorems on convergence of real sequences and series follow through for the complex case (we must replace any modulus notation in the real case by the absolute value of the appropriate complex quantity). Often results can be proved simply by considering the sequence or series as a sum of real and imaginary parts, and applying the real result to each part. Particular examples of results for complex series include:

1. If $\sum_n a_n$ converges then (i) $|a_n| \to 0$ as $n \to \infty$; and (ii) $\exists\ M > 0$ such that $|a_n| < M\ \forall\ n$. (i) may be proved by noting that since the series converges (to s say) then given $\epsilon > 0$ $\exists N$ such that, for all $n \geq N$, $|s_n - s| < \epsilon$ (where s_n denotes the partial sum of the series to n terms. Thus, choosing $m, n \geq N$, we have

   $$|s_m - s_n| = |s_m - s - (s_n - s)| \leq |s_m - s| + |s_n - s| < 2\epsilon \quad \forall m, n \geq N.$$

   Let $m = n + 1$, then the above gives

   $$|s_{n+1} - s_n| \equiv |a_{n+1}| < 2\epsilon \quad \forall n \geq N.$$

   Recalling that $\epsilon > 0$ was arbitrary, this proves the convergence of $|a_n|$ to zero.

2. "Every bounded sequence has a convergent subsequence." This well-known result from real analysis (a variant of the Bolzano-Weierstrass theorem) follows for the series $\Re(z_m)$, giving a convergent subsequence $\Re(z_{m_k})$. The real sequence $\Im(z_{m_k})$ is then another bounded real sequence with a convergent subsequence $\Im(z_{m_{k_j}})$. The complex subsequence $z_{m_{k_j}}$) is then convergent.

3. "Absolute convergence of a series implies convergence." This is a familiar result from real analysis, which has the obvious complex analog, that is, convergence of the series $\sum_n |a_n|$ guarantees convergence of the series $\sum_n a_n$. (Note that the first series here is **real**, while the second is complex.) Proving this result is a fairly straightforward extension of the proof for the real case, noting that $\sum_n a_n$ is essentially the sum of two real sequences:

   $$\sum_n a_n = \sum_n \Re(a_n) + i \sum_n \Im(a_n).$$

4. (Comparison test.) If we have a convergent series with non-negative real terms, $\sum_n b_n$, and a positive constant k such that $|a_n| \leq kb_n$ $\forall n$, then $\sum_n a_n$ converges. Armed with the result 3 above, this is a straightforward extension of the comparison test for real functions - the real result gives convergence of $\sum_n |a_n|$, and then this absolute convergence implies the convergence of the original series $\sum_n a_n$.

Example 3.4 *The series associated with the sequence $a_n = z^n$ is convergent for $|z| < 1$. The partial sum $S_n = \sum_{m=0}^{n} z^n$ is given by*

$$S_n = \frac{1 - z^{n+1}}{1 - z} \quad z \neq 1,$$

from which it is clear that for $|z| < 1$ $S_n \to 1/(1 - z)$, while for $|z| \geq 1$ the series diverges.

(The partial sum S_n in this example may be obtained by noting that $S_{n+1} = zS_n + 1$, and also $S_{n+1} = S_n + z^{n+1}$.)

Homework: Let $z_0 = p \in \mathbb{C}$ and

$$z_{n+1} = \frac{1}{2}(z_n - \frac{1}{z_n}), \quad n \geq 1,$$

if $z_n \neq 0$. Prove the following:
(i) If z_n converges to a limit a then $a^2 + 1 = 0$.
(ii) If $p \in \mathbb{R}$ then z_n, if defined, does not converge.
(iii) If $p = iq$, $0 \neq q \in \mathbb{R}$, then z_n converges.
(iv) If $|p| = 1$ and $p \neq \pm 1$ then z_n converges.

The notion of series leads naturally to the definition of complex **power series**, of the general form

$$\sum_{n=0}^{\infty} c_n z^n, \qquad c_n \in \mathbb{C}. \tag{17}$$

Here the coefficients c_n are understood to be fixed complex numbers, while z may vary. The **radius of convergence** R of the power series is defined by

$$R = \sup\{|z| : \sum |c_n z^n| \quad \text{converges}\}$$

(R may be infinite). The results listed above guarantee that the power series converges within the **circle of convergence** $|z| < R$. To see this, let $|z| < R$. By the definition of R as supremum of the convergence set, there exists some $w \in \mathbb{C}$, with $|z| < |w| \leq R$, such that $\sum |c_n w^n|$ converges. Since $|c_n z^n| < |c_n w^n|$, result 4 above guarantees convergence of $\sum_n c_n z^n$.

Lemma 3.5 *The power series (17) has radius of convergence*

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

where this limit exists.

Proof Since the radius of convergence is defined in terms of the **real** series $\sum |c_n z^n|$, we can use results from real analysis. The **ratio test** for the real series $\sum_{n=0}^{\infty} a_n$ with positive terms states that if

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists then for $L < 1$ the series converges, while if $L > 1$ then the series diverges. Applying this result to $\sum |c_n z^n|$, we have convergence for

$$\lim_{n\to\infty} \left| \frac{c_{n+1} z^{n+1}}{c_n z^n} \right| < 1,$$

and divergence for

$$\lim_{n\to\infty} \left| \frac{c_{n+1} z^{n+1}}{c_n z^n} \right| > 1.$$

Thus, we have convergence for

$$|z| < \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|,$$

and divergence for $|z| > \lim_{n\to\infty} |c_n/c_{n+1}|$. The lemma is proved.

Example 3.6 *The radius of convergence of the power series $\sum_{n=0}^{\infty} z^n$ is 1.*

This can be determined either from a trivial application of lemma 3.5, or directly, as was done in example 3.4.

Example 3.7 *The radius of convergence of the power series* $\sum_{n=0}^{\infty} a^n z^n$*, where* $a \in \mathbb{C}$ *is nonzero, is* $R = 1/|a|$.

This is just a straightforward extension of the example above, and again may be obtained by either applying lemma 3.5 or by the direct method.

Lemma 3.8 *The power series (17) has radius of convergence*

$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}},$$

where this limit exists.

Proof We again use known results from real analysis. The root test (Cauchy's root test) for the real series with positive terms, $\sum_{n=0}^{\infty} a_n$, states that if

$$C = \limsup_{n\to\infty} (a_n)^{1/n}$$

exists (possibly infinite), then if $C < 1$, $\sum_{n=0}^{\infty} a_n$ converges, while if $C > 1$ the series diverges. Applying this result to $\sum_{n=0}^{\infty} |c_n z^n|$ we have convergence if

$$\limsup_{n\to\infty} |c_n z^n|^{1/n} = |z| \limsup_{n\to\infty} |c_n|^{1/n} < 1,$$

and divergence if the inequality is reversed. The result follows.

Example 3.9 *The power series* $\sum a^{n^2} z^n$ *for fixed* $a \in \mathbb{C}$*, with* $|a| > 1$*, does not converge for any* $z \in \mathbb{C}$*, and thus has zero radius of convergence.*

To see this, note that with $c_n = a^{n^2}$,

$$\limsup_{n\to\infty} |c_n|^{1/n} = \limsup_{n\to\infty} (|a^{n^2}|)^{1/n} = \limsup |a|^n = \infty.$$

Hence $R = 0$.

Exercise: (1) Try showing this by other means. (2) Find the radius of convergence of the power series $\sum_n z^n/n!$.

Within the circle of convergence, power series such as (17) can be thought of as defining a **function** of the complex variable z. More generally, a **function** of the complex variable z on a domain D is a mapping or rule f that assigns a unique complex number w to each $z \in D$. We write $w = f(z)$ to represent this mapping, and use the notation $f : D \to \mathbb{C}$ to represent the fact that f takes values in D to other complex values. In general of course if the domain D is only a subset of $\mathbb{C}$, the image $f(D)$ of D under f will only be a subset of $\mathbb{C}$, so you may also see notation $f : D \to f(D)$, or similar. While some real functions require care in their generalization to the complex plane (fractional powers being one example that we consider later), some are straightforward to generalize. For example, integer powers of z are easily generated using the rules of algebra we have already introduced, so that

$$z^2 = (x+iy)(x+iy) = x^2 - y^2 + 2ixy,$$
$$z^3 = (x+iy)(x^2 - y^2 + 2ixy) = x^3 + 3ix^2y - 3xy^2 - iy^3,$$

and so on. From this it is a simple matter to generate complex **polynomial** functions,

$$P(z) = \sum_{n=0}^{N} a_n z^n, \quad a_n \in \mathbb{C};$$

the power series given above are clearly a limiting case of such functions. From polynomials complex **rational** functions are then ratios of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{m=0}^{M} b_m z^m},$$

defined on $\mathbb{C}$ except at those points z^* where $Q(z^*) = 0$.

Exercise: Find the image of the first quadrant, $\Re(z) \geq 0$, $\Im(z) \geq 0$, under the functions $f_1(z) = z^2$, $f_2(z) = z^4$.

In general, since we know $z = x+iy$, and since the function $f(z)$ takes values in $\mathbb{C}$, we can write

$$f(z) = f(x+iy) = u(x,y) + iv(x,y), \tag{18}$$

for real-valued functions u, v, of the two real variables (x, y). We say that u is the **real part** of f, and v is the **imaginary part**.

The **exponential function** can be generalized using the relation (3) that we have already seen. Thus,

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

We note, for future reference, that the absolute value of this complex exponential depends only on the real part of z:

$$|e^z| = |e^x e^{iy}| = e^x.$$

We also have

$$e^{iz} = e^{-y+ix} = e^{-y}(\cos x + i\sin x), \quad e^{-iz} = e^{y-ix} = e^{y}(\cos x - i\sin x). \tag{19}$$

There is a "natural" way to motivate the generalization of the **trigonometric functions** to the case of complex argument. We define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{20}$$

This makes sense, because of the relation (3) for real θ noted earlier, which gives

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

so that (20) is the natural generalization when θ is replaced with a complex number z. The definitions (20) satisfy the usual trigonometric addition formulae for $\sin(z+w)$, *etc.*, and the identity $\cos^2 z + \sin^2 z = 1$ is also easily verified to hold. Other trigonometric functions ($\tan z$, $\cot z$, $\sec z$, $\operatorname{cosec} z$) can then be defined as in the real case, in terms of sine and cosine.

Complex hyperbolic functions can be defined similarly from their real counterparts, as

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2} \tag{21}$$

(from which the other hyperbolic functions can be defined); and again, the addition formulae generalize from the real case as we would expect.

From the definitions (20) and (21) we note that sin and sinh, cos and cosh, are simply related by

$$\sinh iz = i\sin z, \quad \sin iz = i\sinh z, \quad \cosh iz = \cos z, \quad \cos iz = \cosh z.$$

We can also check the consistency of the definitions using the $x + iy$ representation and (19), which gives

$$\frac{e^{iz} - e^{-iz}}{2i} \quad = \quad \frac{1}{2i}(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x))$$

$$
\begin{aligned}
&= \frac{1}{2i}(-\cos x(e^y - e^{-y}) + i\sin x(e^y + e^{-y}) \\
&= i\cos x \sinh y + \sin x \cosh y \\
&= \cos x \sin iy + \sin x \cos iy \\
&= \sin(x + iy) \\
&= \sin z.
\end{aligned}
$$

Remark Note that intuition from real-valued functions does not usually help us with their complex analogs. For example, though the real sine and cosine functions are bounded in absolute value by 1, the complex trigonometric functions are unbounded. If $z = iy$ for $y \in \mathbb{R}$ then $\sin z = i\sinh y$, $\cos z = \cosh y$, so that both $\sin z$ and $\cos z$ are unbounded as $y \to \infty$. Likewise, though the real hyperbolic cosine is always real and positive (always greater than or equal to 1, in fact), the complex hyperbolic cosine has infinitely many zeros in the complex plane – note that for $z = iy$, $\cosh z = \cos y$, and thus the complex hyperbolic cosine has zeros along the imaginary axis at points $z = (n + 1/2)\pi$, $n \in \mathbb{Z}$.

3.2 Limits and continuity of functions

Having seen a few important examples of complex functions, we now return to the general case and develop several of the concepts familiar from real analysis. The first important notion we explore for complex functions is that of continuity, which, as in the real case, may be defined in terms of limits.

Definition 3.10 *(Limit of a function) Let the function f be defined in a neighborhood of z_0 (though possibly not at z_0 itself). Then,*

$$\lim_{z \to z_0} f(z) = w_0$$

(that is, the limit of $f(z)$ as $z \to z_0$ exists and equals w_0) if, $\forall\ \epsilon > 0$ (sufficiently small) $\exists\ \delta > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \textit{whenever} \quad 0 < |z - z_0| < \delta. \tag{22}$$

If z_0 happens to be a boundary point of the region D on which f is defined (or close to the boundary), then (22) should be interpreted to hold only when $z \in D$.

Pictorially, we require the value of the function f to approach w_0 when z approaches z_0 from an **arbitrary direction** in D. The definition 3.10 may be extended to the case $z_0 = \infty$ simply by considering $|z|$ arbitrarily large; thus

$$\lim_{z \to \infty} f(z) = w_0 \quad (|w_0| < \infty)$$

if, given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad |z| > \frac{1}{\delta}.$$

The proofs of the "algebra of limits" results applicable to real-valued functions may be adapted and shown to follow through for complex valued functions.

Homework: (1) Evaluate the following limits, if they exist: (i) $\lim_{z\to 0} \sin z/z$; (ii) $\lim_{z\to\infty} \sin z/z$; (iii) $\lim_{z\to\infty} z^2/(3z+1)^2$; (iv) $\lim_{z\to\infty} z/(z^2+1)$.
(2) If $|g(z)| < M$ for all z in a neighborhood of z_0, and if $\lim_{z\to z_0} f(z) = 0$, show that $\lim_{z\to z_0} f(z)g(z) = 0$.

Definition 3.11 *(Continuity) The function $f(z) : D \to \mathbb{C}$ is said to be* **continuous** *at the point $z_0 \in D$ if $\lim_{z\to z_0} f(z) = f(z_0)$. Alternatively: f is continuous at $z_0 \in D$ if, given $\epsilon > 0$, $\exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.*

We note that f is continuous if and only if both its real and imaginary parts are continuous. **Exercise: Try showing this.**

Example 3.12 *Show that the functions $f(z) = z$ and $f(z) = \overline{z}$ are continuous everywhere in $\mathbb{C}$, and that $g(z) = z^2$ is continuous on any bounded domain.*

The choice $\delta = \epsilon$ works for both cases $f(z) = z, \overline{z}$ here, since, if $|z - z_0| < \epsilon$ then

$$|f(z) - f(z_0)| = |z - z_0| < \epsilon.$$

For $g(z)$ we note that points on a bounded domain satisfy $|z| < M$ for some fixed M. Then, if $|z - z_0| < \delta$ we have

$$|g(z) - g(z_0)| = |z^2 - z_0^2| = |z - z_0||z + z_0| < \delta|z + z_0| \leq \delta(|z| + |z_0|) < 2M\delta,$$

where we used (8). Thus, choosing $\delta = \epsilon/(2M)$, the continuity is proved.

We record one more important theorem on continuous complex functions, that will be needed later:

Theorem 3.13 *(Continuous functions on compact sets) Let $S \subset \mathbb{C}$ be compact, and $f : S \to \mathbb{C}$ continuous. Then f is bounded and attains its bounds; i.e. $\exists z_1, z_2 \in S$ such that*

$$|f(z_1)| \leq |f(z)| \leq |f(z_2)| \quad \forall z \in S.$$

Since the theorem pertains only to the magnitude of f, a real quantity, the proof can proceed as in the real case.

3.3 Differentiability of complex functions

The next important concept is that of **differentiability** of complex functions.

Definition 3.14 *(Differentiable function) Let D be an open set in $\mathbb{C}$. $f : D \to \mathbb{C}$ is* **differentiable** *at $a \in D$ if*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{23}$$

exists (independently of how the limit $h \to 0$, $h \in \mathbb{C}$ is taken). When the limit exists, it gives the **derivative** *of f at a:*

$$f'(a) = \left.\frac{df}{dz}\right|_{z=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If, as is usually the case, the function is differentiable in a **region** then the function is said to be analytic.

Definition 3.15 *(Analytic function) A function $f : D \to \mathbb{C}$ is* **analytic** *at $a \in D$ if f is differentiable in a neighborhood of a. f is analytic on D if it is analytic at every point in D. Analytic functions are also sometimes referred to as* **holomorphic** *functions.*

Analyticity is a very powerful property, and leads to many strong results, as we shall see. It can be shown (from Cauchy's integral formula; see (54) later) that if a function f is analytic then its derivatives of **all** orders exist in the region of analyticity, and all these derivatives are themselves analytic. So, analytic $\Rightarrow$ infinitely differentiable. Differentiability of f at a point z_0 evidently guarantees continuity of f at z_0:

Lemma 3.16 *If $f(z)$ is differentiable at $z_0 \in \mathbb{C}$ then f is continuous at $z_0 \in \mathbb{C}$.*

Proof Differentiability of f at z_0 means that

$$\epsilon_h := f'(z_0) - \frac{f(z_0+h) - f(z_0)}{h} \to 0 \quad \text{as } h \to 0.$$

Then

$$f(z_0+h) - f(z_0) = h(f'(z_0) - \epsilon_h) \to 0 \quad \text{as } h \to 0.$$

Remark At first sight it appears quite hard to write down non-differentiable complex functions f, and one might think that most functions one could write down are differentiable, and even analytic. This seems at odds with the statement often made in textbooks that analytic functions are very special and rare. However, if one writes $z = x + iy$, and thinks of a general complex function f as being composed of real and imaginary parts that are functions of x and y, $f(z) = u(x, y) + iv(x, y)$, then the rarity becomes more apparent. To extract the representation in terms of the complex variable z we have to substitute for $x = (z + \bar{z})/2$ and $y = -i(z - \bar{z})/2$, so that in fact f is, in general, a function of **both z and its complex conjugate $\bar{z}$.** Only if f turns out to be **purely a function of** z can it be analytic.

Definition 3.17 *(Singular point) A point z_0 where the complex function $f(z)$ fails to be analytic is a* **singular point** *of the function.*

We can show explicitly that the function $f(z) = \overline{z}$ (strictly speaking perhaps we ought to write $f(z, \overline{z})$ here, given the remark above) is nowhere analytic. To do this, it suffices to show that different choices of $h \to 0$ give different results. Considering the definition of the derivative, (23), the derivative of this function, if it exists at the point $z \in \mathbb{C}$, is given by

$$f'(z) = \lim_{h \to 0} \frac{\overline{z} + \overline{h} - \overline{z}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}.$$

This limit must exist independently of how the limit is taken, for all possible paths $h \to 0$ in $\mathbb{C}$. Writing $h = |h|e^{i\theta}$, where θ is an arbitrary argument of h, we find

$$f'(z) = \lim_{h \to 0} e^{-2i\theta}.$$

This is clearly not a well-defined limit, since it depends explicitly on the way $h \to 0$ (θ can take **any** real value here). So, the complex conjugate function is not differentiable **anywhere** – though it is continuous everywhere.

Example 3.18 *Likewise, the function $f(z) = |z|^2$ is not differentiable anywhere in $\mathbb{C}$ except the origin.*

This follows since for $z \neq 0$,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)(\overline{z+h}) - z\bar{z}}{h} = \lim_{h \to 0} \frac{z\bar{h} + h\bar{z}}{h}.$$

(For $z = 0$ the limit exists and equals zero.) Again writing $h = |h|e^{i\theta}$ for $0 < |h| \ll 1$ we find that for $z \neq 0$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{|h| \to 0} \frac{|h|ze^{-i\theta} + |h|\bar{z}e^{i\theta}}{|h|e^{i\theta}} = ze^{-2i\theta} + \bar{z},$$

which depends on the way we take the limit $h \to 0$. Note however that $|z|^2 = x^2 + y^2$ is very well-behaved – infinitely differentiable, in fact – as a real valued function in $\mathbb{R}^2$.

These examples illustrate the remark above, since in both cases f is a function of both z and $\bar{z}$, and cannot be expected to be differentiable. For each of these functions, all points in $\mathbb{C}$ are singular points. (Even though $f(z) = |z|^2$ is differentiable at $z = 0$, it is not differentiable in any neighborhood of zero, and therefore not analytic at $z = 0$.)

Example 3.19 *The function $f(z) = z^2$ is differentiable for all $z \in \mathbb{C}$ because*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \to 0} \frac{2zh + h^2}{h} = 2z$$

independently of how $h \in \mathbb{C}$ approaches zero.

More generally:

Example 3.20 *The function $f(z) = z^n$ is differentiable, with $f'(z) = nz^{n-1}$. If n is a positive integer (or zero) then this result holds $\forall\, z \in \mathbb{C}$, but if n is a negative integer then f is not differentiable at the point $z = 0$.*

In fact the result holds also for arbitrary powers of z, but since we have not yet learned how to deal with non-integer powers we restrict to integers for now. To prove the result for $n \in \mathbb{N}$ we use the standard binomial expansion for $(z+h)^n$ (knowing how to evaluate integer powers of complex numbers, this may be proved by induction as in the real case), giving

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{(z+h)^n - z^n}{h} \\ &= \lim_{h\to 0} \frac{z^n + nhz^{n-1} + n(n-1)h^2 z^{n-2}/2! + \cdots + h^n - z^n}{h} \\ &\to nz^{n-1} \quad \text{as } h \to 0. \end{aligned}$$

Almost the same proof works for the case of negative integers n if we assume that the binomial expansion for this case is valid for complex numbers (it is). For fixed $z \neq 0$ we simply extract the factor z^n from $(z+h)^n$ and then expand $(1 + h/z)^n$ for $|h/z| < 1$. We obtain the same result as $h \to 0$.

Homework: (1) Show from first principles that the function $f(z) = 1/z$ is (a) continuous; (b) analytic in $\mathbb{C} \setminus B(0;r)$ for any $r > 0$. (The point $z = 0$ is the only singular point of this function.)
(2) Show from first principles that, if f is continuous, nonzero and differentiable at $z_0 \in \mathbb{C}$, then $1/f$ is differentiable at z_0, and its derivative at this point is given by $-f'(z_0)/f(z_0)^2$. [NOTE: You are being asked to **prove** this form of the quotient rule here, not apply it!]
(3) Find where the following functions are differentiable (and hence analytic): (i) $\sin z$; (ii) $\tan z$; (iii) $(z-1)/(z^2+1)$; (iv) $\Re(z)$.
You may assume that the sum and product of two functions is differentiable on any region where the two functions are themselves differentiable. Any results shown in lectures may be assumed.
(4) Let $f(z)$ be continuous everywhere. Show that, if $f(z_0) \neq 0$, then there exists a neighborhood of z_0 in which $f(z) \neq 0$.

3.4 The Cauchy-Riemann theorem

Differentiability (or non-differentiability) of a complex function $f(z)$ may also be established by appealing to the **Cauchy-Riemann theorem.**

Theorem 3.21 *(Cauchy-Riemann) The function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$ of $D \subset \mathbb{C}$ if and only if the partial derivatives u_x, u_y, v_x, v_y are continuous and satisfy the* **Cauchy-Riemann equations**

$$u_x = v_y, \qquad u_y = -v_x \tag{24}$$

in a neighborhood of z.

Proof Suppose that $f = u + iv$ is differentiable at the point z. Then $f'(z)$ exists, however we take the limit $h \to 0$ in (23). Thus we let $h \to 0$ with $h \in \mathbb{R}$ to obtain

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h, y) - u(x, y) + i(v(x+h, y) - v(x, y)}{h} \\ &= u_x(x, y) + iv_x(x, y). \end{aligned} \tag{25}$$

Alternatively, since we know the result is independent of how we take the limit, we let $h \to 0$ with $h = i|h|$ pure imaginary. Then

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+i|h|) - f(z)}{i|h|} \\ &= \lim_{|h|\to 0} \frac{u(x, y+|h|) - u(x, y) + i(v(x, y+|h|) - v(x, y)}{i|h|} \\ &= -iu_y(x, y) + v_y(x, y). \end{aligned} \tag{26}$$

These two results must be identical for a well-defined limit and thus

$$u_x = v_y, \quad u_y = -v_x.$$

Conversely, we suppose now that the first-order partial derivatives of u and v exist; are continuous; and satisfy the Cauchy-Riemann equations (24). We consider taking the limit of a small variation from the point $z = x+iy$, which we can represent variously as

$$z \to z + \delta z = x + iy + \delta x + i\delta y.$$

This variation induces changes $\delta u = u(x+\delta x, y+\delta y) - u(x,y)$, $\delta v = v(x+\delta x, y+\delta y) - v(x,y)$, $\delta f = f(z+\delta z) - f(z)$, in u, v and f, which, by results from real analysis (Taylor's theorem), satisfy

$$\begin{aligned}
\delta u &= \delta x u_x(x,y) + \delta y u_y(x,y) + \epsilon_u, \\
\delta v &= \delta x v_x(x,y) + \delta y v_y(x,y) + \epsilon_v,
\end{aligned}$$

where $\epsilon_u/|\delta z|$, $\epsilon_v/|\delta z| \to 0$ as $\delta z \to 0$. Thus, in the definition of the derivative,

$$\begin{aligned}
\frac{\delta f}{\delta z} &= \frac{\delta u + i\delta v}{\delta x + i\delta y} \\
&= \frac{\delta x u_x + \delta y u_y + i(\delta x v_x + \delta y v_y) + \epsilon_u + i\epsilon_v}{\delta x + i\delta y} \\
&= \frac{u_x(\delta x + i\delta y) - iu_y(\delta x + i\delta y) + \epsilon_u + i\epsilon_v}{\delta x + i\delta y} \\
&= u_x - iu_y + \frac{\epsilon_u + i\epsilon_v}{\delta z} \\
&\to u_x - iu_y \quad \text{as } \delta z \to 0,
\end{aligned}$$

where we used the Cauchy-Riemann equations (24) and the limiting behavior of ϵ_u and ϵ_v in the above. Thus, the derivative of f at the point z exists, and the proof is complete.

Note 1 The Cauchy-Riemann equations show that, for an analytic function $f(z) = u(x,y) + iv(x,y)$, we have $f'(z) = u_x + iv_x = v_y - iu_y$ (equations (25) and (26)). Higher derivatives of f, when they exist, can also be computed in terms of higher derivatives of the real and imaginary parts, noting that, if $g(z) = f'(z) = u_x + iv_x$ then u_x and v_x are (respectively) the real and imaginary parts of g and thus

$$g'(z) = f''(z) = (u_x)_x + i(v_x)_x = u_{xx} + iv_{xx}.$$

Exercise: It is clear that there are several possible expressions for $f''(z)$. Find them all.

Note 2 The Cauchy-Riemann equations also confirm the remark made above, that only if f turns out to be purely a function of z can it be analytic. To see this we note that

$$x = \frac{1}{2}(z + \bar{z}), \quad y = -\frac{i}{2}(z - \bar{z}) \quad \Rightarrow \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

(using the chain rule for partial differentiation). Thus, if u and v satisfy the Cauchy-Riemann equations then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x + iu_y + i(v_x + iv_y)) = \frac{1}{2}(u_x - v_y + i(u_y + v_x)) = 0,$$

so that f is independent of $\bar{z}$.

It is becoming clear that the real and imaginary parts of an analytic function are closely related, and that one cannot hope to choose real-valued functions $u(x, y)$ and $v(x, y)$ independently, stick them together as real and imaginary parts, and obtain an analytic function. The choice of real part $u(x, y)$ almost uniquely determines the imaginary part v (we will make this statement precise soon).

Returning to example 3.18 above, with $f(z) = |z|^2$ we have $u(x, y) = x^2 + y^2$, $v(x, y) = 0$. The Cauchy-Riemann equations are not satisfied in the neighborhood of any $z \in \mathbb{C}$, confirming the above finding that this function is nowhere analytic.

Homework: Ablowitz & Fokas, problems for section 2.1, questions 1,2.

3.4.1 Harmonic functions

An immediate consequence of the Cauchy-Riemann theorem is that the real and imaginary parts of an analytic function are **harmonic**.

Definition 3.22 *(Harmonic function) Any function $u(x, y)$ with continuous 2nd derivatives satisfying*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{27}$$

is **harmonic.** *Equation (27) is* **Laplace's equation.**

Lemma 3.23 *The real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ are harmonic.*

We are anticipating a result that will be established later, that analyticity of f guarantees that the second derivatives (in fact, derivatives of all orders) of its real and imaginary parts exist and are continuous on the domain D on which f is defined and analytic.

Proof Since f is analytic its derivatives of all orders exist, and thus the partial derivatives of u and v exist and are continuous at all orders in the domain of analyticity. By the Cauchy-Riemann equations we then have

$$u_{xx} + u_{yy} = \frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial x}(v_y) + \frac{\partial}{\partial y}(-v_x) = v_{xy} - v_{xy} = 0.$$

Similarly we can show that $v_{xx} + v_{yy} = 0$, so that both u and v are harmonic.

We will later prove a converse (lemma 3.24 below) to this theorem: that harmonic functions can be used to construct analytic functions.

Lemma 3.24 *If $u(x, y)$ is a harmonic function on a simply-connected domain D then a* **harmonic conjugate** *v exists such that u and v satisfy the Cauchy-Riemann equations (24), and $f = u + iv$ is an analytic function on D.*

Example 3.25 *The analytic function $f(z) = z^4$ has real and imaginary parts $u(x, y) = x^4 - 6x^2y^2 + y^4$, $v(x, y) = 4x^3y = 4xy^3$, both of which are harmonic everywhere. We know this without doing any differentiation, by lemma 3.23. It's also easily checked that u and v satisfy the Cauchy-Riemann equations.*

3.5 Multivalued functions

We have seen that many "real" functions such as polynomials, rational functions, exponential, trigonometric and hyperbolic functions, generalize in a relatively straightforward way to the complex case. However, when we think about inverting some of these functions (we often need to be able to invert analytic functions), it is clear that in some cases the inverse function should be able to map a complex number to several values. We are already familiar with this from real analysis, where we know that taking the square-root of a positive number gives two values: the equation $y^2 = x > 0$ has two solutions, $y = \pm\sqrt{x}$, for any real positive number x. Likewise, real-valued inverse trigonometric functions are also multivalued. In the complex case also we would expect to be able to obtain two solutions to the equation

$$w^2 = z, \tag{28}$$

for any given $z \in \mathbb{C}$, but how do we interpret $\sqrt{z}$? Likewise (though this does not arise for the real case), we saw that the exponential function e^w can return the same value for many different choices of w (for example, all values $w_n = i(\theta + 2n\pi)$ for any integer n give the same result for e^w). Thus a sensible inverse for the complex exponential function (which is a complex version of the logarithm) should be a multi-valued function.

We begin by considering how one can define a complex square-root (solving equation (28) for $w(z)$). If we try to follow the route of taking real and imaginary parts things get quite cumbersome. Writing $z = x+iy$, $w = u+iv$, we find

$$u^2 - v^2 = x, \quad 2uv = y;$$

substituting for v from the second of these into the first gives a real quartic equation for u – four possible solutions (but hard to find), and some may (will!) be spurious when substituted back into the original equation $w^2 = z$. There is a much better way to find solutions to equations such as (28), and it involves writing the complex number z, whose square-root (or any other fractional power for that matter) we wish to take, in the complex exponential form (3). Equation (28) is then

$$w^2 = |z|e^{i\theta} = |z|e^{i(\theta_0 + 2n\pi)},$$

where θ is any argument of the complex number z and θ_0 is a specific choice (which will usually be taken as the principal argument in applications). Then, as in the real case, we set

$$w = f(z) = |z|^{1/2} e^{i(\theta_0/2 + n\pi)} \tag{29}$$

(where $|z|^{1/2}$ is interpreted as in the real case, since $|z| \in \mathbb{R}$). Again, this relation holds for any integer n, but in fact w can only take two distinct values:

$$n = 0: \qquad\qquad w = |z|e^{i\theta_0/2}, \tag{30}$$

$$n = 1: \quad w = |z|^{1/2} e^{i(\theta_0/2+\pi)} = -|z|^{1/2} e^{i\theta_0/2}. \tag{31}$$

Other integer values of n give these same values (even values of n give the same as $n = 0$, and odd n the same as $n = 1$). These two distinct definitions of $z^{1/2}$ are known as **branches** of the square-root function $f(z)$.

3.5.1 General non-integer powers

We can define other non-integer powers of z in the same way, noting that

$$w^\alpha = z \quad \Rightarrow \quad w = z^{1/\alpha} = |z|^{1/\alpha} e^{i(\theta_0/\alpha + 2n\pi/\alpha)}, \quad n \in \mathbb{N}. \tag{32}$$

The simplest case is the mth root function ($\alpha = m \in \mathbb{N}$), which has m distinct branches, defined by

$$w = f(z) = z^{1/m} = |z|^{1/m} e^{i(\theta_0/m + 2n\pi/m)}, \quad n = 0, 1, \ldots, m-1. \tag{33}$$

In the case that α is a rational number, $\alpha = m/j$ say, for integers j, m (with no common factors), there are only m distinct values of the right-hand side in (32), giving m distinct branches of the function. If α is irrational then there are infinitely many distinct branches.

3.5.2 Logarithms

The second major class of multifunctions is given by the complex logarithm. As mentioned, this function is obtained by inverting the complex exponential, $z = e^w$. Writing z in its complex exponential form again, and $w = u + iv$, we see that

$$e^w = e^{u+iv} = e^w e^{iv} = |z| e^{i(\theta_0 + 2n\pi)},$$

so that $e^u = |z|$ (giving $u = \ln|z|$) and $v = \theta_0 + 2n\pi$. Thus, we can interpret the inverse function, the complex logarithm $w = \ln z$, as

$$w = u + iv = \ln|z| + i(\theta_0 + 2n\pi) = \ln z \tag{34}$$

where θ_0 is any choice from the set of values $\arg(z)$ (again, the principal argument of z is usually the most sensible definition of the complex logarithm). As above, the different forms of the function that arise from taking different values of n in (34) are known as the **branches** of the complex logarithm. There are always infinitely many of them.

It is easily verified that the usual algebraic rules for logarithms, such as $\log(z_1 z_2) = \log z_1 + \log z_2$, etc., are satisfied in the complex case also (where distinct branches of the logarithms are understood; if we have the n_1 branch of $\log z_1$ and the n_2 branch of $\log z_2$, then in general we obtain the (n_1+n_2)-branch of $\log(z_1 z_2))$. To see this we note that $z_1 z_2 = |z_1 z_2| e^{i(\theta_1 + 2n_1\pi + \theta_2 + 2n_2\pi)}$ (with obvious notation), so that

$$\begin{aligned}
\ln(z_1 z_2) &= \ln(|z_1||z_2|) + i(\theta_1 + 2n_1\pi + \theta_2 + 2n_2\pi) \\
&= \ln|z_1| + i(\theta_1 + 2n_1\pi) + \ln|z_2| + i(\theta_2 + 2n_2\pi) \\
&= \ln(z_1) + \ln(z_2),
\end{aligned}$$

using the real result for $\ln(|z_1||z_2|)$ in the second line. Also, we note that for $z = |z|e^{i(\theta_0 + 2n\pi)}$ we can define a multifunction

$$\begin{aligned}
\ln(1/z) &= \ln((1/|z|)e^{-i(\theta_0+2n\pi)}) = \ln(1/|z|) - i(\theta_0 + 2n\pi) \\
&= -\ln|z| - i(\theta_0 + 2n\pi) = -\ln z.
\end{aligned} \tag{35}$$

Application of these results enables us to show that in general, $\ln(z_1/z_2) = \ln(z_1) - \ln(z_2)$.

3.5.3 Contours and multivaluedness

From the definitions (29), (33), (34), it is clear that as we traverse any circle $z = re^{i\theta}$ (with r fixed and θ going from 0 to 2π), although we return to the starting-point in the z-plane when $\theta = 2\pi$, any chosen branch of the multifunction $f(z)$ does not return to its original value. To see this explicitly, we look first at the simplest example of a multi-valued function, the square-root $f(z) = z^{1/2}$ defined in (29), and consider how either branch of this function changes as z moves along the circular contour. The first branch of the square-root function, (29) with $n = 0$, is just

$$\sqrt{z} = |z|^{1/2} e^{i\theta/2} = \begin{cases} |z|^{1/2} & \theta = 0, \\ -|z|^{1/2} & \theta = 2\pi^-. \end{cases}$$

Thus, by traversing the contour and returning to the original point, we have moved from one branch of the function (30) to the other (31) (clearly, this branch as defined is discontinuous across the positive real axis). Likewise, if we start with the $n = 1$ branch (31) of the square-root and allow z to traverse the same circular contour then we see that this branch has

$$\sqrt{z} = -|z|^{1/2} e^{i\theta/2} = \begin{cases} -|z|^{1/2} & \theta = 0, \\ |z|^{1/2} & \theta = 2\pi^-, \end{cases}$$

and we move from the $n = 1$ branch back to the $n = 0$ branch.

Similar results hold for the other multifunctions. If we start with the n-branch of the mth root function (33) then on the circular contour $|z| = r$ we have

$$\begin{aligned} z^{1/m} &= |z|^{1/m} e^{i(\theta/m + 2n\pi/m)} \\ &= \begin{cases} |z|^{1/m} e^{2in\pi/m} & \theta = 0, \\ |z|^{1/m} e^{i(2\pi/m + 2n\pi/m)} = |z|^{1/m} e^{2i(n+1)\pi/m} & \theta = 2\pi^-, \end{cases} \end{aligned}$$

so here we move from the n to the $(n+1)$-branch. m complete traversals of the contour will return us to the original branch that we started with. For the complex logarithm (34), traversal of the same contour would again take us from the n to the $(n + 1)$-branch of the function; but in this case, no matter how many times we traverse the contour we will never return to the branch we started with.

In general, for functions with a finite number of branches (like the mth root function), repeated traversals of the circular contour always eventually brings us back to the original value of the function, once we have cycled through all the branches. For functions with infinitely many branches though, such as irrational powers or the complex logarithm, we will never return to the original value.

We can see that for the multifunctions defined so far, any closed contour that encloses the origin $z = 0$ will lead to the same result, since the argument of z does not return to its original value, but increases by 2π. For any closed contour that does **not** enclose the origin, $\arg(z)$ **does** return to its original value after a complete circuit of the contour. It is clear that the point $z = 0$ has a special status, since it is only closed contours that encircle this point that lead to a jump in the value of the function. We say that $z = 0$ is a **branch-point** of these multifunctions. More generally:

Definition 3.26 *(Branch-point) Let $w = f(z)$ be a multifunction defined on $S \subset \mathbb{C}$. $z = a \in \mathbb{C}$ is a* **branch point** *of the multifunction $f(z)$ if, for all sufficiently small $r > 0$, it is not possible to choose a single branch of f that defines a continuous function on $C(a; r)$ (the circular contour with center a and radius r).*

We shall see the need (below) to restrict to "sufficiently small" values of r. The point $z = \infty$ is also often a branch point for the function. To check whether this is the case we need to modify the above definition slightly:

Definition 3.27 *(Branch-point at infinity) We say $z = \infty$ is a branch point of the multifunction $f(z)$ if the point 0 is a branch point of the multifunction $\hat{f}(z) = f(1/z)$.*

It is clear from these definitions that both 0 and infinity are branch points of the multifunctions (29), (32), (33) and (34) (to see that ∞ is a branch point of (34) use (35)).

Example 3.28 *$z = \infty$ is a branch-point of the mth root function $f(z) = z^{1/m}$ for integers $m \geq 2$.*

We need to show that $z = 0$ is a branch-point of $\hat{f}(z) = f(1/z)$. With (as usual) $z = |z|e^{i(\theta_0+2n\pi)}$, then noting that $1/z = (1/|z|)e^{-i(\theta_0+2n\pi)}$, we have the nth branch of $\hat{f}(z) = f(1/z)$ given by

$$\hat{f}_n(z) = |z|^{-1/m}e^{-i(\theta_0/m+2n\pi/m)}.$$

Traversing a circular contour $|z| = r$ (constant), then as θ_0 increases from 0 to 2π we have

$$\hat{f}_n(z) = \begin{cases} |z|^{-1/m}e^{-2in\pi/m} & \theta_0 = 0, \\ |z|^{-1/m}e^{-2i(n+1)\pi/m} & \theta_0 = 2\pi. \end{cases}$$

Clearly the branch is not continuous on the contour (we have moved from the n to the $(n+1)$-branch), and so $z = 0$ is a branch point of $\hat{f}(z)$, thus $z = \infty$ is a branch point of f.

Note: It is easily checked that if z is replaced by $(z - a)$ in any of the preceding examples (29)–(34) then the resulting functions have branch-points at $z = a$ and $z = \infty$.

3.5.4 Branch cuts to ensure single-valuedness

Often in applications it is necessary to ensure single-valuedness of a function; and the above discussion suggests that the way to ensure this for the functions defined so far is to prevent any means by which z could make a complete circuit of the origin. The least restrictive way to do this (for these particular multifunctions) is to **cut** the z-plane from the origin to infinity, outlawing any circuits that cross the cut. Often several choices of cut are possible, but in most cases the cut will be made along the real axis, either along $[0, +\infty)$ or along $(-\infty, 0]$. Such cuts in the complex plane are known as **branch cuts**, since they effectively make individual branches of the multifunctions single-valued. Normally a single branch of the multifunction suffices in applications; a convenient choice is often the principal branch (defined using the principal argument). Most of the multifunctions we shall consider are not only single-valued in the cut plane, but are also analytic there.

Example: The square-root function. We cut the plane along the real positive axis, and work with the principal argument, which is constrained to lie in the interval $0 \leq \theta < 2\pi$. This argument function is continuous in the cut plane, since points of the cut are excluded. The principal branch of the square-root function is then just the $n = 0$ branch (30):

$$f(z) = |z|^{1/2} e^{i\theta/2},$$

taking values $|z|^{1/2}$ on the top side of the cut, $\theta = 0^+$, and $-|z|^{1/2}$ on the bottom side of the cut, $\theta = 2\pi^-$. [The $n = 1$ branch (31) is also single-valued in this cut plane, the values taken on the upper and lower sides of the cut being reversed from the $n = 0$ case.]

The discussion above suggests that branch cuts will always be taken between branch points, so as to outlaw contours that encircle a branch point (by definition 3.26, any such contour would lead to a discontinuity in f). There is usually a certain amount of freedom in the choice of branch cuts, and convenience dictates how we choose them in a given situation. It is fairly straightforward to define suitable branch cuts in the simple examples of multifunctions discussed so far, but we shall now consider more complicated multifunctions for which the choice is not so obvious.

3.6 "Composite" multifunctions

§3.5 showed how to deal with elementary examples of multivalued functions, in which the branch-points and suitable branch-cuts are easily identified. We now consider how to define more complicated functions, with several branch points in the finite complex plane. Again the procedure is best illustrated by example.

3.6.1 Example: Product of square-roots

We consider first a generalization of the square-root function,

$$f(z) = [(z-a)(z-b)]^{1/2}, \quad a, b \in \mathbb{C}. \tag{36}$$

We can write each of the complex numbers $(z-a)$, $(z-b)$ in polar form,

$$z - a = r_a e^{i\theta_a + 2in_a\pi}, \quad z - b = r_b e^{i\theta_b + 2in_b\pi},$$

where $r_a = |z-a|$ and θ_a is (without loss of generality) the principal argument of $(z-a)$, lying in the interval $0 \le \theta_a < 2\pi$; and similarly for r_b, θ_b. The multifunction (36) can then be defined by

$$f(z) = [(z-a)(z-b)]^{1/2} = r_a^{1/2} r_b^{1/2} e^{i(\theta_a+\theta_b)/2 + in\pi} \quad \begin{cases} n = n_a + n_b \in \mathbb{Z}, \\ 0 \le \theta_{a,b} < 2\pi. \end{cases} \tag{37}$$

As with (29) only the values $n = 0, 1$ give distinct results (other values of n repeat these), so there are just two distinct branches. It is clear that if we let z vary along any simple closed contour γ_a that surrounds just a and not b then, over one complete circuit of γ_a, θ_a changes by 2π while θ_b returns to its original value. Thus, the quantity $i(\theta_a + \theta_b)/2$ appearing in the exponent in (37) has a net change of $i\pi$, and so $f(z)$ moves from one branch to the other as we traverse γ_a once. In the same way, if z traverses a simple closed contour γ_b that encloses b and not a, then θ_b changes by 2π over one complete circuit, while θ_a returns to its original value. **However**, if z makes one complete circuit of a simple closed contour γ_{ab} that encloses both a and b then, since both θ_a and θ_b change by an integer multiple of 2π, the quantity $(\theta_a + \theta_b)/2$ appearing in the definition (37) changes by an integer multiple of 2π also. Thus in this case $f(z)$ returns to its original value on completion of the circuit.

Looking at the definition of branch-points again, it is clear that both a and b are branch points for the function (37). To check the point at infinity, note that

$$\tilde{f}(z) = f(1/z) = [(1/z - a)(1/z - b)]^{1/2} = \frac{1}{z}[(1-az)(1-bz)]^{1/2},$$

which, although it is singular at $z = 0$, does not have a branch point there. So $z = \infty$ is not a branch point of (37).

The above discussion about contours suggests that we should take a branch-cut between a and b in the z-plane, since a cut here prevents z varying continuously along contours such as γ_a and γ_b, but allows z to make excursions that encircle both a and b. Again, a cut may in principle be made along any curve joining the two points, but the simplest choice is to cut along the straight line $[a, b]$.

Looking at the definition (37), we can now check that such a branch-cut leads to a single-valued function. For simplicity in this checking we shall look at the case $a = -1$, $b = 1$, where the branch cut runs along $[-1, 1]$ in the z-plane. We take the $n = 0$ branch and restrict θ_a and θ_b to be the principal arguments:

$$f_0(z) = [(z+1)(z-1)]^{1/2} = r_{-1}^{1/2} r_1^{1/2} e^{i(\theta_{-1}+\theta_1)/2}, \quad 0 \le \theta_{-1}, \theta_1 < 2\pi.$$

For most of the cut plane it is evident that $f_0(z)$ is continuous and single-valued, because θ_{-1} and θ_1 are. The definition is only problematic as z crosses the (allowed part of) the positive real axis, $z \in \mathbb{R}^+$, $|z| > 1$, where θ_{-1} and θ_1 have jump discontinuities. Just above the positive real axis, where $\arg(z) = 0^+$ (and $|z| > 1$) we have $\theta_{-1} = \theta_1 = 0$; so here

$$f_0(z) = r_{-1}^{1/2} r_1^{1/2}.$$

Just below this part of the positive real axis, where $\arg(z) = 2\pi^-$, we have $\theta_{-1} = \theta_1 = 2\pi$; so here

$$f_0(z) = r_{-1}^{1/2} r_1^{1/2} e^{i(2\pi+2\pi)/2} = r_{-1}^{1/2} r_1^{1/2}.$$

Thus, as defined by (38), $f(z)$ is continuous in the cut plane.

For the $n = 1$ branch we have

$$f_1(z) = [(z+1)(z-1)]^{1/2} = r_{-1}^{1/2} r_1^{1/2} e^{i(\theta_{-1}+\theta_1)/2+i\pi} = -f_0(z), \quad 0 \le \theta_{-1}, \theta_1 < 2\pi,$$

so this branch also will be continuous along any contour that encloses both 1 and -1.

Exercise: Check specifically that for this example neither branch of f is continuous across the cut.

3.6.2 Example: Logarithm of a product

We next consider a logarithmic function with a more complicated argument:

$$f(z) = \log((z-a)(z-b)). \tag{38}$$

By the rules of logarithms we know this is equivalent to $\log(z-a)+\log(z-b)$, so we anticipate branch points at both $z=a$, $z=b$, and perhaps at $z=\infty$. To check explicitly, as above we write $z-a = r_a e^{i\theta_a + 2in_a\pi}$, where without loss of generality $\theta_a \in [0, 2\pi)$ is the principal argument of $(z-a)$ and n_a is an arbitrary integer; and similarly for $(z-b)$. Then

$$\begin{aligned} f(z) &= \log(r_a r_b e^{i(\theta_a+\theta_b+2n\pi)}), \quad n = n_a + n_b \in \mathbb{Z} \\ &= \log(r_a r_b) + i(\theta_a + \theta_b + 2n\pi). \end{aligned}$$

In this case, any (simple closed) contour for which a complete circuit gives a discontinuity in **either** θ_a or θ_b, will also give a discontinuity in $f(z)$. Thus we must cut the plane so as to prevent z making a complete circuit around either or both of the points a or b. The way to do this is to make two cuts, one from a to ∞, and the other from b to ∞. The fact that our cuts must extend to infinity suggests that this is another branch point, and we can show this explicitly by considering

$$\begin{aligned} \tilde{f}(z) &= f(1/z) = \log((1/z-a)(1/z-b)) = \log((1-az)(1-bz)/z^2) \\ &= \log(1-az) + \log(1-bz) - 2\log z. \end{aligned}$$

Because of the $\log z$ term, a small circuit around $z=0$ (that does not also enclose $z=1/a$ or $z=1/b$) will lead to a discontinuity in $\arg(z)$ when a complete circuit is made, and hence a discontinuity in $f(1/z)$, showing that $z=\infty$ is indeed also a branch point of this function.

For the specific case $a=-1$, $b=1$, suitable branch-cuts are from the point $z=-1$ to infinity, along the negative real axis, and from the point $z=1$ to infinity, along the positive real axis.

3.7 Summary and concluding remarks on multifunctions

In the discussion above, we were firstly concerned with how to define the multi-valued functions in question in a sensible and consistent way. This led to definitions of functions with several (possibly infinitely many) **branches**, characterized by an integer n. Each branch corresponds to a different selection of argument of the complex variable, each choice of argument being restricted to an interval of length 2π.

This led us to think about how to define a **single-valued** (analytic) branch of the function. In each example considered we chose the simplest branch (usually the principal $n = 0$ branch defined via the principal argument(s)), though we could, of course, use any of the branches to construct a single-valued function. Single-valuedness is ensured by introducing appropriate **branch cuts** in the complex z-plane, which prevent z from moving in the plane in any manner that would lead to a discontinuity in f. Only circuits around certain points in the z-plane can induce such discontinuities; such points are known as **branch-points** of the function (definitions 3.26 and 3.27).

We saw that if we do not restrict the plane by branch-cuts, then f is discontinuous along certain lines in the z-plane. Mathematically, as z crosses such lines, we move from one branch of the function to another. An alternative way of rationalizing the structure of multifunctions is to suppose that in fact a multifunction is defined on several "copies" of the complex plane, one for each branch. As z encircles a branch point, we move from one copy of the plane to another (for the function $\log z$, for example, this complex plane would be something like a helix: see figure 2.3.6 in [1]). Such a multi-layered complex plane is known as a Riemann surface, with each complete copy of $\mathbb{C}$ being a single **sheet** of this surface. For the square-root function the Riemann surface is more analogous to a Mobius strip; two complete circuits of the Riemann surface bring us back to our starting point.

Homework Ablowitz & Fokas, problems for Section 2.3, questions 1, 2, 3. Where the question does not demand this explicitly, you should also define the branches of the multifunctions concerned.

4 Complex Integration

With the ideas of **paths** through complex space introduced in §§2.4 and 2.4.1, and our notions of complex functions, we are now in a position to integrate complex-valued functions along paths in the complex plane.

4.1 Basic methods of complex integration

Complex functions may be integrated along contours in the complex plane in much the same way as real functions of 2 real variables can be integrated along a given curve in (x, y)-space. An integration may be performed directly, by introducing a convenient parametrization of the contour; or by splitting into real and imaginary parts, each of which can be evaluated as real line integrals. Also, as we shall soon see, integrals can often be evaluated by appealing to one of several powerful theorems of complex analysis.

Example 4.1 *(Direct integration by parametrization) Evaluate*

$$I = \oint_C \left(\frac{a}{z} + bz\right) dz \quad a, b \in \mathbb{C},$$

where C is the circle $|z| = r$ in $\mathbb{C}$.

This contour is conveniently parametrized by the real variable $\theta \in [0, 2\pi]$ as $C = \{z \in \mathbb{C} : z = re^{i\theta}\}$, so that $dz = ire^{i\theta} d\theta = iz\, d\theta$. Then

$$I = \int_0^{2\pi} i(a + br^2 e^{2i\theta})\, d\theta = 2\pi ia + \frac{br^2}{2}[e^{2i\theta}]_0^{2\pi} = 2\pi ia.$$

The nonzero result here is a reflection of the fact that the function f has a singularity within the contour of integration at $z = 0$. Note also that the result is the same regardless of the radius of the circular contour. We will see from **Cauchy's theorem** (see §5 later) that any function **analytic** on a domain contained within a **closed** integration contour would give a zero result when integrated around the contour; likewise, for a function like this that contains just one singularity, the result is the same for any simple closed contour that encloses the singularity.

In principle one could also evaluate complex contour integrals by writing $z = x + iy$, $dz = dx + idy$ and evaluating the real and imaginary parts along the contour of integration. In the example above this would be very cumbersome, and parametrization is clearly the best option. We now consider an example where it is best to stick with the cartesian representation.

Example 4.2 *(Splitting into real and imaginary parts) Evaluate (a)* $\int_\gamma \overline{z}\, dz$*, (b)* $\int_\gamma z\, dz$*, where* γ *is (i) the straight line joining* $z_0 = 0$ *to* $z_1 = 1+i$*; (ii) the parabolic contour* $\Im(z) = \Re(z)^2$ *joining the same two points in the complex plane.*

(a(i)) We write the integral as real and imaginary parts,

$$I = \int_\gamma \overline{z}\, dz = \int_\gamma (x - iy)(dx + idy) = \int_\gamma xdx + ydy + i(xdy - ydx),$$

and then note that $x = y$ on γ, with x running from 0 to 1. Thus the imaginary part vanishes, and we are left with

$$I = \int_\gamma \overline{z}\, dz = 2\int_0^1 xdx = 1.$$

(a(ii)) On the second curve $y = x^2$ and so the above result gives the integral as

$$\begin{aligned} I &= \int_\gamma xdx + ydy + i(xdy - ydx) = \int_0^1 (x + 2x^3)\, dx + i\int_0^1 (2x^2 - x^2)\, dx \\ &= 1 + i/3. \end{aligned}$$

(b(i)) As above we have,

$$I = \int_\gamma z\, dz = \int_\gamma (x + iy)(dx + idy) = \int_\gamma xdx - ydy + i(xdy + ydx),$$

and with $x = y$ on γ, with $0 \le x \le 1$ the real part vanishes and

$$I = \int_\gamma z\, dz = 2i\int_0^1 xdx = i.$$

(b(ii)) Now $y = x^2$ and so the above gives

$$\begin{aligned} I &= \int_\gamma xdx - ydy + i(xdy + ydx) = \int_0^1 (x - 2x^3)\, dx + i\int_0^1 (2x^2 + x^2)\, dx \\ &= i. \end{aligned}$$

Observe that in case (a) we get a different result for the two contours, hence a nonzero result when we integrate $\overline{z}$ around the simple closed contour made by joining the two curves together. In case (b) however the two integration curves give the same result, and hence integrating z around the simple closed contour made by joining the curves will give zero – again, this is due to the fact that z is a complex analytic function while $\overline{z}$ is nowhere analytic.

Example 4.3 *(Example 4.1 revisited) Replace the integrand in example 4.1 by $a/\overline{z} + bz$:*

$$I = \oint_{C(0;r)} \left(\frac{a}{\overline{z}} + bz \right) dz.$$

Parametrizing as before, we have

$$\begin{aligned} I &= \int_0^{2\pi} \left(\frac{a}{re^{-i\theta}} + bre^{i\theta} \right) ire^{i\theta} \, d\theta \\ &= \int_0^{2\pi} i(a + br^2)e^{2i\theta} \, d\theta = \frac{1}{2}(a + br^2)[e^{2i\theta}]_0^{2\pi} = 0. \end{aligned}$$

The integrand here is nowhere analytic (firstly as it is a function of $\overline{z}$ as well as z, and secondly because it is unbounded at the origin), yet still integrates to zero around any circular contour centered on the origin. When a function is nowhere analytic it is often difficult to make predictions about its properties. However, this example shows that, just because a function happens to integrate to zero around a family of simple closed curves, we cannot deduce that it is analytic. The converse to Cauchy's theorem requires a stronger condition (Morera's theorem 6.6).

More generally, to integrate a complex function $f(z)$ along a contour $\gamma \in \mathbb{C}$ using parametrization:

$$\gamma = \{z(t) : t \in [a, b]\}, \tag{39}$$

we have

$$\int_\gamma f(z) \, dz = \int_a^b f(z(t))z'(t) \, dt. \tag{40}$$

This last representation is equivalent to a line integral along a curve in the (x, y)-plane, where the function to be integrated takes complex values.

Alternatively, for a general function $f(z) = u(x, y) + iv(x, y)$ we have

$$\int_\gamma f(z)\,dz = \int_\gamma (u(x, y)dx - v(x, y)dy + i(u(x, y)dy + v(x, y)dx), \tag{41}$$

where on the right-hand side γ is interpreted as a curve in (x, y)-space.

Homework: Ablowitz & Fokas, problems for section 2.4, questions 1,2,3,4.

4.2 General results for complex integration

Reversing the direction of integration changes the sign of the result If the contour γ is followed in the opposite direction, which is denoted by $-\gamma$, then (40) gives

$$\int_{-\gamma} f(z)\,dz = \int_b^a f(z(t))z'(t)\,dt = -\int_a^b f(z(t))z'(t)\,dt = -\int_\gamma f(z)\,dz.$$

Integrals along piecewise smooth contours If γ consists of sub-contours γ_k joined (continuously, but not necessarily smoothly) together, then

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{N} \int_{\gamma_k} f(z)\,dz. \tag{42}$$

Theorem 4.4 *(Fundamental theorem of calculus) Let $F(z)$ be analytic in a domain D, and $f(z) = F'(z)$ be continuous in D. Then for a piecewise smooth contour γ lying in D, with endpoints z_1 and z_2, we have*

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1). \tag{43}$$

Proof We assume first that the contour is smooth, with a parametrization of the form (39). Then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_\gamma F'(z)\,dz \\ &= \int_a^b F'(z(t))z'(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \frac{d}{dt}(F(z(t)))\,dt \\
&= F(z(b)) - F(z(a)) \\
&= F(z_2) - F(z_1)
\end{aligned}
$$

(we go from the third line to the fourth line by applying the real Fundamental theorem of calculus to the real functions $\Re(F(z(t)))$, $\Im(F(z(t))))$.

If the contour γ is only piecewise smooth the result still follows by using (42) and following through the above steps for each sub-curve γ_k. Contributions from the "internal" endpoints cancel.

Two consequences are immediate from theorem 4.4:

Corollary 4.5 *For any* **closed** *contour γ lying within the domain of analyticity D of the function F, if $f = F'$ on D, we have*

$$\int_\gamma f(z)\,dz = 0.$$

This result is a special case of **Cauchy's theorem**.

Remark Note that it is certainly not the case in general for real-valued functions $u(x, y)$, even when perfectly smooth, that the real line integral around a simple closed contour in $\mathbb{R}^2$ vanishes. For example, integrating the function $u(x, y) = x^2 + y^2$ around the contour $x^2 + y^2 = 1$, we have (transforming to polar coordinates in which $ds = d\theta$ on γ):

$$\oint_\gamma (x^2 + y^2)\,ds = \oint_0^{2\pi} 1\,d\theta = 2\pi.$$

Nor is it the case that an arbitrary smooth real-valued function integrated with respect to z around a closed contour vanishes. For example, with the same integration contour as above, and $u(x, y) = x(x^2 + y^2)$ we have

$$
\begin{aligned}
\oint_\gamma x(x^2 + y^2)\,dz &= \int_0^{2\pi} \cos\theta . ie^{i\theta}\,d\theta \quad \text{(parametrizing the contour)} \\
&= i\pi.
\end{aligned}
$$

Corollary 4.6 *Let $f(z) = F'(z)$ for $F(z)$ analytic on a domain $D \subset \mathbb{C}$. If γ_1 and γ_2 are two (piecewise smooth) curves lying in D, with the same endpoints z_1 and z_2, then*

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz,$$

that is, the result of integrating between two given points in D is **path-independent**.

We note that this last result does **not** hold for the example 4.2(a), since the function $f(z) = \overline{z}$ is nowhere analytic. However it does hold for 4.2(b).

Homework: Check that if $\overline{z}$ is replaced by z^2 in example 4.2 that the same result is obtained for both choices of contour joining the points $z = 0$ and $z = 1$.

In many applications (such as inverting Laplace or Fourier transforms, for example), it is not necessary to evaluate a contour integral explicitly, but only find a suitable bound for it (often one wishes to know that a certain integral tends to zero as a limiting form of the contour is taken – a common example is the need to show that an integral of a function along a circular contour of radius R (or ϵ) goes to zero as $R \to \infty$ (or $\epsilon \to 0$)).

Theorem 4.7 *(Estimation theorem for complex integrals) Suppose γ is a path with parametrization (39), and that the function $f(z)$ is continuous on γ. Then*

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_a^b |f(z(t))z'(t)|\ dt.$$

Proof (Theorem 4.7) We know that for a real-valued function $g(t)$ integrable on $[a, b]$

$$\left| \int_a^b g(t)\, dt \right| \leq \int_a^b |g(t)|\, dt. \tag{44}$$

We have

$$\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(z(t))z'(t)\, dt \right| = e^{i\phi} \int_a^b f(z(t))z'(t)\, dt$$

for some $\phi \in \mathbb{R}$. Then

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b \Re[e^{i\phi} f(z(t))z'(t)]\,dt\right|.$$

Applying the real inequality (44) with $g(t) = \Re[e^{i\phi} f(z(t))z'(t)]$, we find

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &\leq \int_a^b |\Re[e^{i\phi} f(z(t))z'(t)]|\,dt \\ &\leq \int_a^b |e^{i\phi} f(z(t))z'(t)|\,dt \\ &\leq \int_a^b |f(z(t))z'(t)|\,dt.\end{aligned}$$

Corollary 4.8 *Suppose γ is a path with parametrization (39), and that the function $f(z)$ is continuous on γ, with $|f(z)| \leq M \forall z \in \gamma$. Then*

$$\left|\int_\gamma f(z)\,dz\right| \leq M \int_a^b |z'(t)|\;dt \equiv ML(\gamma).$$

Here, $L(\gamma) = \int_a^b |z'(t)|\;dt$ is the **length** of the contour. It is easily checked that this definition of contour length agrees with that for the length of a real curve in $\mathbb{R}^2$. Simply switch from the given parametrization t to arclength parametrization s, with respect to which γ is specified by $(x(s), y(s))$, with endpoints at $s = s_a$ and $s = s_b$. Then note that $(dz/dt)dt = (dz/ds)ds$, so that

$$L(\gamma) = \int_{s_a}^{s_b} \sqrt{x_s^2 + y_s^2}\,ds = \int_{s_a}^{s_b} |z'(s)|\,ds = \int_a^b |z'(t)|\,dt.$$

The proof of corollary 4.8 is trivial. The inequality is also often written as

$$\left|\int_\gamma f(z)\,dz\right| \leq M \int_a^b |dz|,$$

since $dz = z'(t)dt$, and dt is real and positive.

Example 4.9 *Show that $\oint_\gamma (z^2+1)^{-1}\,dz \to 0$ as $R \to \infty$, where γ is the circular contour of radius R centered on the origin.*

Note: This circular contour arises frequently, and we will usually denote it by $C(0;R)$. More generally, the circle with center a and radius r is denoted $C(a;r)$.

Points $z(t)$ on the contour are given by $z(t) = Re^{it}$, $0 \le t < 2\pi$, so that the estimation theorem 4.7 gives

$$\left|\int_\gamma \frac{dz}{z^2+1}\right| \le \int_0^{2\pi}\left|\frac{iRe^{it}}{R^2e^{2it}+1}\right| dt \le \int_0^{2\pi}\frac{R}{R^2-1}dt = \frac{2\pi R}{R^2-1} \to 0 \,\text{as}\; R\to\infty.$$

Example 4.10 *Let $\phi(z)$ be a function such that $|z\phi(z)| \to 0$ as $|z| \to \infty$, and let γ_N be the square contour whose sides are $\Re(z) = \pm N$, $\Im(z) = \pm N$, for $N \in \mathbb{N}$. Show that*

$$\int_{\gamma_N} \phi(z)\tan\pi z\,dz \to 0 \quad \textit{as } N\to\infty.$$

[Results such as this can be useful in calculating infinite sums, as we will see later.]
We split the contour into its four straight-line segments: γ_1: $\Re(z) = N$, $-N < \Im(z) < N$; γ_2: $\Im(z) = N$, $-N < \Re(z) < N$; γ_3: $\Re(z) = -N$, $-N < \Im(z) < N$; γ_4: $\Im(z) = -N$, $-N < \Re(z) < N$, and consider each segment separately.

Our strategy here is to bound $\tan\pi z$ on each part of the contour, which is not trivial since, as we now know, the sine and cosine function are not bounded in the complex plane. On γ_1 $z = N + iy$, and

$$\begin{aligned}
|\tan\pi z| &= \left|\frac{\sin\pi z}{\cos\pi z}\right| = \left|\frac{\sin\pi(N+iy)}{\cos\pi(N+iy)}\right| \\
&= \left|\frac{\sin N\pi\cos i\pi y + \cos N\pi \sin i\pi y}{\cos N\pi\cos i\pi y - \sin N\pi\sin i\pi y}\right| \\
&= \left|\frac{\sin i\pi y}{\cos i\pi y}\right| = \left|\frac{\sinh\pi y}{\cosh\pi y}\right| = |\tanh\pi y| \le 1.
\end{aligned}$$

Similarly on γ_3, $|\tan\pi z| \le 1$.

On γ_2, $z = x + iN$ and

$$|\tan\pi z| = \left|\frac{\sin\pi z}{\cos\pi z}\right| = \left|\frac{\sin\pi(x+iN)}{\cos\pi(x+iN)}\right|$$

$$
\begin{aligned}
&= \left|\frac{\sin \pi x \cos i\pi N + \cos \pi x \sin i\pi N}{\cos \pi x \cos i\pi N - \sin \pi x \sin i\pi N}\right| \\
&= \left|\frac{\sin \pi x \cosh \pi N + i \cos \pi x \sinh \pi N}{\cos \pi x \cosh \pi N - i \sin \pi x \sinh \pi N}\right| \\
\Rightarrow \quad |\tan \pi z|^2 &= \frac{\sin^2 \pi x \cosh^2 \pi N + \cos^2 \pi x \sinh^2 \pi N}{\cos^2 \pi x \cosh^2 \pi N + \sin^2 \pi x \sinh^2 \pi N} \\
&= \frac{\cosh^2 \pi N - \cos^2 \pi x}{\cosh^2 \pi N - \sin^2 \pi x} \\
&\le \frac{\cosh^2 \pi N}{\cosh^2 \pi N - 1}.
\end{aligned}
$$

Clearly this is bounded for large values of N (it becomes arbitrarily close to 1 as N increases) – in fact, for $N \ge 1$ it is always less than 1.01! Similarly, $\tan \pi z$ is bounded on γ_4. So, we can say that

$$|\tan \pi z| \le M \quad \text{on } \gamma_N,$$

for some fixed $M > 0$. The condition on the function $\phi(z)$ gives

$$|z\phi(z)| \le \epsilon_N \quad \text{on } \gamma_N,$$

where $0 < \epsilon_N \to 0$ as $N \to \infty$. Since $|z| \ge N$ on γ_N, we then have

$$|\phi(z)| \le \frac{\epsilon_N}{N} \quad \text{on } \gamma_N.$$

We can now apply the estimation theorem to deduce that

$$\left|\int_{\gamma_N} \phi(z) \tan \pi z \, dz\right| \le \sup_{\gamma_N} |\phi(z) \tan \pi z| \int_{\gamma_N} |dz| \le \frac{M\epsilon_N}{N} L(\gamma_N) = 8M\epsilon_N$$

and hence the integral goes to zero as $N \to \infty$, as claimed.

Note that the choice of contour in this example is very important. We could not have taken a square of arbitrary size, because we must avoid the singularities of the integrand at points $z_n = (n + 1/2)\pi$. We certainly could not bound $\tan \pi z$ on a contour that passes through any such points.

Homework Ablowitz & Fokas, problems for section 2.4, questions 7, 8.

4.3 Harmonic functions to construct analytic functions

We now recall Lemma 3.24 stated, but not proved, earlier (and also our comments when considering the Cauchy-Riemann theorem, that the real and imaginary parts of a complex analytic function are closely related). With our knowledge of complex integration we are now in a position to prove the result, restated below:

Lemma 4.11 *If $u(x, y)$ is a harmonic function on a simply-connected domain D then a* **harmonic conjugate** *v exists such that u and v satisfy the Cauchy-Riemann equations (24), and $f = u + iv$ is an analytic function on D.*

The proof relies on the well-known result:

Theorem 4.12 *(Green's theorem in the plane) Let $p(x, y)$, $q(x, y)$ be real-valued functions, with continuous partial derivatives in a simply-connected region S made up of the interior of a simple closed contour γ (described in the usual anticlockwise sense), and the contour itself. Then*

$$\oint_\gamma (pdx + qdy) = \int\int_S \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy.$$

The proof of this theorem can be found in any textbook on vector calculus.

Proof (of lemma 4.11). We find a harmonic conjugate v by construction. Let $(x_0, y_0) \in D$ and set $v(x_0, y_0) = 0$. Define the value of v at other points $(x, y) \in D$ by the following integral, taken along some path within D joining (x_0, y_0) to (x, y):

$$v(x, y) = \int_{(x_0,y_0)}^{(x,y)} -u_y \, dx + u_x \, dy.$$

This is a good definition provided the value of the integral is independent of the path taken within D from (x_0, y_0) to (x, y). Suppose C_1 and C_2 are two such different paths within D, then the integrals along these two paths are the same if the integral around the closed curve C, made by joining C_1 and $(-C_2)$, is zero. But for any closed curve C we have

$$\oint_C -u_y \, dx + u_x \, dy = \int\int_S \left(\frac{\partial}{\partial x}(u_x) - \frac{\partial}{\partial y}(-u_y) \right) dxdy, \qquad (45)$$

where S is a spanning surface within D for the closed curve C (this is where the simple-connectedness is required), by Green's theorem in the plane; and the right-hand side of (45) vanishes because u is harmonic on D. Thus a conjugate function v can always be defined in this way; and by construction u and v satisfy the Cauchy-Riemann equations

$$v_y = u_x, \qquad v_x = -u_y,$$

so that $u + iv$ defines an analytic function on D by theorem 3.21.

5 Cauchy's theorem

5.1 Cauchy's theorem I

We shall now prove the main result concerning complex integration, from which many other useful theorems follow. There are many different ways of proving this theorem; in general the fewer restrictions we impose in the theorem statement, the more technical the proof required. We shall give the most general theorem statement later, and for now restrict ourselves to proving a more basic version.

Theorem 5.1 *(Cauchy) Let $f(z)$ be analytic in a simply-connected domain D. Then for any simple closed contour γ in D,*

$$\int_\gamma f(z)\, dz = 0.$$

Later we will be able to remove the restrictions on simple connectedness of D, and on simpleness of γ, provided (loosely speaking) that γ does not encircle any "holes" in D.

The proof given below relies on a key result from real analysis (Green's theorem in the plane, stated as theorem 4.12 above) and follows that given in Ablowitz & Fokas [1]. Alternative proofs are possible that are more closely allied with results from **topology**, and that give stronger versions of the theorem.

Before proving Cauchy's theorem, we remark that the proof given below using Green's theorem "requires" also that $f'(z)$ be continuous in the domain D of the theorem statement. In fact this property is automatic for any analytic function – but we have not proved this yet! (This proof of the existence and continuity of (all) higher derivatives of f relies itself on Cauchy's theorem.) It is possible to prove Cauchy's theorem without the assumption of continuity on f', but the proof is more technical.

Proof (Cauchy's theorem) We use the result (41),

$$\oint_\gamma f(z)\,dz = \oint_\gamma (udx - vdy) + i\oint_\gamma (udy + vdx).$$

With $f'(z)$ continuous, u and v also have continuous partial derivatives (see (25), (26)), thus Green's theorem 4.12 is applicable, and

$$\oint_\gamma f(z)\,dz = -\int\int_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i\int\int_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy.$$

Analyticity of f means that the Cauchy-Riemann equations (24) hold, and hence the theorem is proved.

Example 5.2 *Evaluate the integral of $f(z) = z^n$ around the unit circle $|z| = 1$.*

By Cauchy's theorem we know that the result must be zero for integers $n \geq 0$, because f is analytic then. We can show the result directly by parametrizing the contour by $z = e^{i\theta}$, $dz = ie^{i\theta}$, then

$$\begin{aligned} \oint_\gamma f(z)\,dz &= \int_0^{2\pi} e^{in\theta} ie^{i\theta}\,d\theta = i\int_0^{2\pi} (\cos(n+1)\theta + i\sin(n+1)\theta)d\theta \\ &= i\left[\frac{\sin(n+1)\theta}{n+1}\right]_0^{2\pi} + \left[\frac{\cos(n+1)\theta}{n+1}\right]_0^{2\pi} = 0, \quad n \neq -1. \end{aligned}$$

So in fact the result is zero for a wider range of n than predicted by Cauchy's theorem. This was easy to demonstrate for the chosen circular contour, but would be more difficult if we chose (e.g.) the square contour in the next example. We will soon prove another theorem (again due to Cauchy) that can predict the above result for all values of n and for all contours (Cauchy's Residue theorem).

Example 5.3 *The integral of $f(z) = \sin z$ around the square contour γ with sides $\Re(z) = 0,\ \pi$, $\Im(z) = 0,\ \pi$, is zero by Cauchy's theorem.*

We could of course have chosen a more complicated contour, but then we would not be able (easily) to verify the result directly. For a direct verification we parametrize each part of the contour separately, writing

$$\begin{aligned}
\gamma_1 &= \{z = \pi + iy : 0 < y < \pi\}, \\
\gamma_2 &= \{z = x + i\pi : \pi > x > 0\}, \\
\gamma_3 &= \{z = 0 + iy : \pi > y > 0\}, \\
\gamma_4 &= \{z = x + i0 : 0 < x < \pi\}.
\end{aligned}$$

Then on γ_1, $dz = idy$, and $f(z) = \sin(\pi + iy) = -\sin(iy) = -i \sinh y$, so

$$\int_{\gamma_1} f(z)\, dz = \int_0^\pi \sinh y\, dy = [\cosh y]_0^\pi = (\cosh \pi - 1).$$

On γ_2, $dz = dx$, and $f(z) = \sin(x + i\pi) = \sin x \cos(i\pi) + \cos x \sin(i\pi) = \cosh \pi \sin x + i \sinh \pi \cos x$, so

$$\begin{aligned}
\int_{\gamma_2} f(z)\, dz &= \int_\pi^0 (\cosh \pi \sin x + i \sinh \pi \cos x)\, dx \\
&= -\cosh \pi [\cos x]_\pi^0 + i \sinh \pi [\sin x]_\pi^0 = -2 \cosh \pi.
\end{aligned}$$

On γ_3, $dz = idy$, and $f(z) = \sin(0 + iy) = i \sinh y$, so

$$\int_{\gamma_3} f(z)\, dz = -\int_\pi^0 \sinh y\, dy = -[\cosh y]_\pi^0 = (-1 + \cosh \pi).$$

On γ_4, $dz = dx$, and $f(z) = \sin x$, so

$$\int_{\gamma_4} f(z)\, dz = \int_0^\pi \sin x\, dx = -[\cos x]_0^\pi = 2.$$

Adding the four contributions, the result is zero, as predicted by Cauchy's theorem.

Homework: Ablowitz & Fokas, problems for section 2.6, questions 1,3,4.

5.2 Cauchy's theorem II

Though the proof given above is adequate for most purposes, it is useful to have a proof that does not assume continuity of $f'(z)$, since this result can actually be deduced from Cauchy's theorem. However, as indicated above, such a proof, though possible, is much more technical, requiring the development of several topological ideas, and the proof of several preliminary results. Since many of these concepts are interesting and useful in their own right however; and since results such as infinite smoothness of f may be deduced from Cauchy's theorem (so it would be nice not to have to assume any smoothness of f' in the proof); we shall investigate how Cauchy's theorem may be proved in more generality.

Our first step is to prove Cauchy's theorem for an arbitrary triangular contour.

Theorem 5.4 *(Cauchy's theorem for a triangle) Let f be analytic inside and on a triangular contour γ, then*

$$\int_\gamma f(z)\,dz = 0.$$

Proof We use the notation ABC to denote the triangle formed by joining points $z = a$, $z = b$, $z = c$, and we suppose γ is the triangle $A_0B_0C_0$. Take points A_1, B_1, C_1 to be the midpoints of edges B_0C_0, A_0C_0, A_0B_0, respectively, from which we can construct 4 more "ABC" triangles ($A_1B_1C_1$, $A_0B_1C_1$, $A_1B_0C_1$ and $A_1B_1C_0$). We denote these 4 triangles by γ_1^k, $k = 1, \dots, 4$ (the "1" to denote that this is the first step in an iterative triangulation process, and the k just as a convenient label for each sub-triangle). Then we have

$$I := \int_\gamma f(z)dz = \sum_{k=1}^{4} \int_{\gamma_1^k} f(z)dz,$$

using (42) (line integrals along interior segments of the sub-triangles cancel). The inequality (8) applied to this result gives

$$|I| \le \sum_{k=1}^{4} \left| \int_{\gamma_1^k} f(z)dz \right| \le 4 \left| \int_{\gamma_1^{k_*}} f(z)dz \right|,$$

where k_* is the choice of k giving the maximal absolute value to the integral. Label this triangle $\gamma_1^{k_*}$ as γ_1, and repeat the above argument, with sub-triangles (γ_2^k) of γ_1. This procedure, illustrated schematically in figure 3,

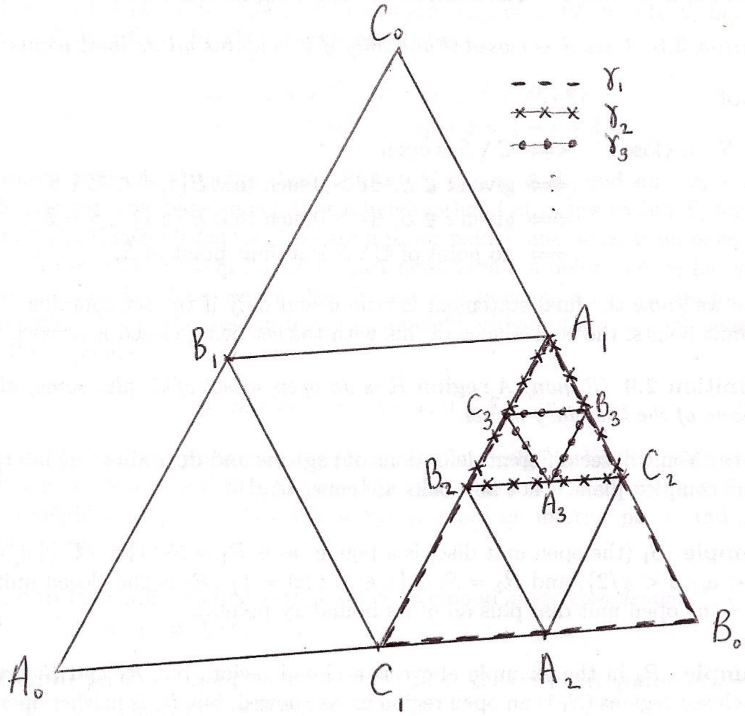

Figure 3: Iterative triangulation of the original contour γ (the outermost triangle), so as to maximize $|\int_{\gamma_k} f(z)\,dz|$ at each step.

generates a sequence of triangles (γ_j) (with $\gamma_0 = \gamma$), such that, for all $j \geq 1$,

$$\begin{array}{ll} \text{(i)} & \overline{I(\gamma_{j+1})} \subset \overline{I(\gamma_j)} \\ \text{(ii)} & L(\gamma_j) = 2^{-j} L(\gamma) \\ \text{(iii)} & 4^{-j}|I| \leq |\int_{\gamma_j} f(z)\,dz|. \end{array}$$

Claim: The set $\cap_{j=0}^{\infty}\overline{I(\gamma_j)}$ contains a point Z common to all $\overline{I(\gamma_j)}$. [Recall that $I(\gamma)$ is the interior of the contour γ, a bounded open set.]
Proof of claim: For each j choose $z_j \in \overline{I(\gamma_j)}$. The resulting sequence (z_j) is bounded, since all points $z_j \in I(\gamma)$. By result 2 of §3.1, we have a convergent subsequence with limit Z. For any j, Z is a limit point of the sequence $\{z_k : k \geq j\} \subset \overline{I(\gamma_j)}$, and so (since $\overline{I(\gamma_j)}$ is closed), Z belongs to $\overline{I(\gamma_j)}$.

Now let $\epsilon > 0$ (arbitrarily small). f is differentiable at Z so, for some $\delta > 0$,

$$|f(z) - f(Z) - (z - Z)f'(Z)| < \epsilon|z - Z| \quad \forall z \in B(Z;\delta). \tag{46}$$

Choose N sufficiently large that $\overline{I(\gamma_N)} \subset B(Z,\delta)$. Then, we have

$$|z - Z| \leq 2^{-N}L(\gamma) \quad \text{for } z \in \overline{I(\gamma_N)}, \text{ by (ii) above,} \tag{47}$$

and

$$\int_{\gamma_N} (f(Z) + (z - Z)f'(Z))dz = 0, \tag{48}$$

by application of theorem 4.4 (recall that Z, $f(Z)$, $f'(Z)$ are all fixed constants here). Hence, using (46)–(48), and the estimation theorem 4.7, we have

$$\begin{aligned} \left|\int_{\gamma_N} f(z)\,dz\right| &= \left|\int_{\gamma_N} f(z)\,dz - \int_{\gamma_N} (f(Z) + (z - Z)f'(Z))dz\right| \quad \text{using (48)} \\ &\leq \int_{\gamma_N} |f(z) - f(Z) - (z - Z)f'(Z))|dz \\ &< \epsilon\int_{\gamma_N} |z - Z|\,dz \quad \text{using (46)} \\ &\leq \epsilon 2^{-N} L(\gamma)L(\gamma_N) \quad \text{using corollary 4.8} \\ &= \epsilon(2^{-N}L(\gamma))^2 \quad \text{using (ii) above.} \end{aligned}$$

Finally we use result (iii) listed above to deduce that

$$|I| \leq 4^N \left| \int_{\gamma_N} f(z)\, dz \right| < \epsilon L(\gamma)^2.$$

Since $\epsilon > 0$ was arbitrary, and $L(\gamma)$ is fixed, $I = 0$ as required.

Remark Cauchy's theorem for polygonal contours: Of course, it follows from this proof that Cauchy's theorem holds for any polygonal contour γ (triangulate the polygon and write the integral of f around γ as a sum of integrals along triangular contours).

In order to extend Cauchy's theorem to more general contours and domains, we need to derive a few results that give conditions under which an analytic function f can be shown to have an antiderivative F. Our first goal is to extend Cauchy's theorem to arbitrary contours γ lying within a convex domain on which f is analytic. The following theorem guarantees that an antiderivative can be found for any f analytic on a convex domain; though in fact in its statement it is somewhat stronger than that.

Theorem 5.5 *(Antiderivative on a convex domain) Let $f(z)$ be continuous on the convex domain D, and suppose that $\int_\gamma f(z)\, dz = 0$ for any triangle $\gamma \in D$. Let $a \in D$ be arbitrary, then*

$$F(z) = \int_{[a,z]} f(w)\, dw$$

is analytic in D, with $F' = f$.

Here the integral is taken along the line segment $[a, z]$. Note that this theorem is immediately applicable to any f analytic on D, since we have just proved that for such f, the integral along arbitrary triangular contours is zero (theorem 5.4).

Proof Let $z \in D$. Since D is open, $\exists \delta > 0$ such that $B(z; \delta) \subset D$. Then for all $|h| < \delta$, $z + h \in D$. For such h, the three line segments $[a, z]$, $[z, z + h]$, $[a, z+h]$, all lie in D (convexity), and form a triangle within D. The integral of f around this triangle must be zero, by the theorem hypothesis. Hence we have

$$\begin{aligned}
F(z+h)-F(z) &= \int_{[a,z+h]} f(w)dw - \int_{[a,z]} f(w)dw \\
&= -\int_{[z+h,a]} f(w)dw - \int_{[a,z]} f(w)dw \\
&= \int_{[z,z+h]} f(w)dw.
\end{aligned}$$

We then have

$$\begin{aligned}
\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &= \frac{1}{|h|}\left|\int_{[z,z+h]} (f(w)-f(z))dw\right| \\
&\leq \frac{1}{|h|}|h| \sup_{w\in[z,z+h]} |f(w)-f(z)|
\end{aligned}$$

by the estimation theorem result 4.8. The right-hand side here goes to zero as $h \to 0$ by continuity; and the theorem is proved.

For convenience, we restate this result specifically for analytic functions (for which it holds automatically as noted above):

Theorem 5.6 *(Antiderivative for analytic function on convex domain) Let f be analytic on a convex domain D. Then $\exists\ F$, analytic on D, such that $F' = f$.*

We can now extend Cauchy's theorem to any convex domain:

Theorem 5.7 *(Cauchy's theorem on a convex domain) Let f be analytic on a convex domain D. Then*

$$\int_\gamma f(z)\,dz = 0$$

for every closed path $\gamma \in D$.

Proof Theorem 5.6 guarantees the existence of an analytic antiderivative F, such that $F' = f$ on D. Then the result follows from the fundamental theorem of calculus (corollary 4.5).

An immediate consequence of theorem 5.7 is that for any function that is complex analytic on a convex domain D, its integral along a path between two points $a, b \in D$ is **path independent** – the result is the same no matter which path within D we choose.

The next version of Cauchy's theorem that we give (theorem 5.10 below) relates to the integral of an analytic function around an arbitrary simple closed contour, and is sufficient for most purposes. Since we cannot assume that we are dealing with contours composed of simple line segments as in the results derived so far, we must first prove an important preliminary result, which we now state:

Theorem 5.8 *(Covering theorem for contours) Let γ be a contour, with parametric description (39), and suppose $\gamma \subset D$ where $D \subset \mathbb{C}$ is open. Then $\exists\ \delta > 0$ (fixed), and open balls $B_k = B(z(t_k); \delta)$, with $k = 0, \ldots, N$ and $a = t_0 < t_1 < \ldots < t_N = b$, such that*
(i) $B_k \cap B_{k+1} \neq \emptyset$, $k = 0, \ldots, N$;
(ii) $\gamma_k := z([t_k, t_{k+1}]) \subset B_k$, $k = 0, \ldots, N$;
(iii) $\gamma \subset \cup_{k=0}^{N} B_k \subset D$.

Informally, this theorem guarantees that when a contour lies inside an open subset $D \subset \mathbb{C}$, it may be entirely covered by a finite sequence of $(N+1)$ overlapping balls, of fixed radius, such that all the balls lie within D. The construction also partitions the contour into N individual segments, each of which is entirely contained within one of the balls.

The proof of theorem 5.8 requires the following Lemma:

Lemma 5.9 *Let D be an open subset of $\mathbb{C}$, and $\gamma \in D$. Then $\exists\ \delta > 0$ such that $\forall z \in \gamma$, $B(z; \delta) \subset D$.*

The informal statement of this lemma is that a contour in an open subset of $\mathbb{C}$ cannot get arbitrarily close to the boundary of D.

Proof of Lemma 5.9. Let $\rho(z)$ be the distance of z from $D^c = \mathbb{C} \setminus D$:

$$\rho(z) = \inf\{|z - w| : w \notin D\}.$$

For points z, $z' \in \mathbb{C}$ and $w \notin D$ we have

$$\begin{aligned} \rho(z) \le |z - w| &\le |z - z'| + |z' - w| \\ \Rightarrow \quad |z' - w| &\ge \rho(z) - |z' - z|. \end{aligned}$$

Taking the infimum over $w \notin D$ we find

$$\rho(z') \geq \rho(z) - |z - z'|.$$

But clearly we can reverse the roles of z and z', so that also

$$\rho(z) \geq \rho(z') - |z - z'|;$$

and so $-|z - z'| \leq \rho(z) - \rho(z') \leq |z - z'|$, that is,

$$|\rho(z) - \rho(z')| \leq |z - z'|, \quad \forall z, z' \in \mathbb{C}.$$

Thus $\rho(z)$ is a continuous function; and on the compact set represented by the contour γ, ρ is bounded and attains its bounds (theorem 3.13). In particular it attains its infimum at some point $z^* \in \gamma$. Then, because D is open and $z^* \in D$, $\exists \delta > 0$ such that $B(z^*; \delta) \subset D$. By definition of the infimum on the set γ, and of the distance function ρ

$$\rho(z) \geq \rho(z^*) \geq \delta \quad \forall z \in \gamma.$$

Again using the definition of the distance function ρ, $\rho(z) \geq \delta \Leftrightarrow B(z; \delta) \subset D$, proving the theorem.

With the lemma in hand, we can prove theorem 5.8.

Proof (of theorem 5.8). Let δ be as in lemma 5.9 above, so that $\forall z \in \gamma$, $B(z; \delta) \subset D$. We have to show that γ can be covered by a finite chain of such disks, each overlapping the next.

Assume that γ is smooth (for piecewise smooth contours the following arguments can be used for each smooth portion of γ). The Mean Value theorem applied to the real and imaginary parts of the parametrization function $z(t)$ (see (39)) gives

$$\Re(z(t)) - \Re(z(s)) = (t - s)\Re((z'(c)) \quad \text{for some } c \in [a, b],$$

with a similar result for $\Im(z(t))$. The continuous real functions $\Re(z'(t))$, $\Im(z'(t))$ are defined on the compact set $[a, b]$, so are bounded (and attain their bounds) on $[a, b]$. Let M be a common bound for these functions. We therefore have

$$|\Re(z(t)) - \Re(z(s))| < \delta/2 \quad \text{whenever} \quad |t - s| < \eta = \delta/(2M),$$
$$|\Im(z(t)) - \Im(z(s))| < \delta/2 \quad \text{whenever} \quad |t - s| < \eta = \delta/(2M),$$

which together imply (using (8)), that

$$|z(t) - z(s)| < \delta \quad \text{whenever} \quad |t - s| < \eta = \delta/(2M).$$

As noted above, this result holds for an **arbitrary** contour γ by applying the argument to its smooth constituent parts.

We now select points $a = t_0 < t_1 < \ldots < t_N = b$, such that $t_{k+1} - t_k < \eta$ for $k = 0, 1, \ldots, N$. Taking $B_k = B(z(t_k), \delta)$, the conditions of the theorem are all satisfied.

We can now prove a yet stronger version of Cauchy's theorem:

Theorem 5.10 *(Cauchy's theorem, version 3) Suppose that f is analytic inside and on a simple closed contour γ. Then*

$$\int_\gamma f(z)\, dz = 0.$$

Proof Note that if γ is a polygonal contour then the result is immediate from Cauchy's theorem for triangular contours (theorem 5.4). As noted earlier, we can simply triangulate the polygon and write

$$\int_\gamma f(z)\, dz = \sum_j \int_{\gamma_j} f(z)\, dz,$$

where the γ_j are the triangulating contours (the integrals along the interior line segments are each covered twice, in the opposite sense, and thus cancel). Since the integral around each γ_j vanishes (theorem 5.4), the result follows.

For an arbitrary contour γ, our approach is to **approximate** γ by a polygonal contour. We use theorem 5.8 to cover γ by overlapping discs $B_k = B(z(t_k); \delta)$, $k = 0, \ldots, N$ satisfying all conditions listed there and, additionally, with $z(t_N) = z(t_0)$ (a closed contour with $z(a) = z(b)$). Within each disk, which is a convex domain, Cauchy's theorem 5.7 applies, giving

$$\int_{\gamma_k} f(z)\, dz = \int_{[z_k, z_{k+1}]} f(z)\, dz,$$

where γ_k is the restriction of the contour γ to the parameter interval $[t_k.t_{k+1}]$, and $[z_k, z_{k+1}]$ is the line segment from $z_k = z(t_k)$ to $z_{k+1} = z(t_{k+1})$. Hence, the total integral along the contour γ is equal to the integral along the closed polygonal contour γ_P made by joining the line segments $[z_0, z_1]$, $[z_1, z_2]$, ..., $[z_{N-1}, z_N]$,

$$\int_\gamma f(z)\, dz = \sum_{k=0}^{N-1} \int_{[z_k, z_{k+1}]} f(z)\, dz = \int_{\gamma_P} f(z)\, dz.$$

The result then follows, by the noted result for polygonal contours.

Remark We note that we require the concept of the **interior** of the contour (introduced in definition 2.20), since it is not possible to triangulate an unbounded domain. Hence the theorem is applicable only to the contour's interior. This appeal to triangulation in the final step is also the reason why the theorem works only for a simply-connected domain – we cannot triangulate a closed polygonal curve on a non-simply-connected domain, in general.

An important consequence of theorem 5.10 is the following contour deformation theorem.

Theorem 5.11 *(Contour deformation) Let γ_1 and γ_2 be positively-oriented closed contours, with $\gamma_2 \subset I(\gamma_1)$, and suppose that f is analytic in the region $(I(\gamma_1) \cap E(\gamma_2)) \cup \{\gamma_1\} \cup \{\gamma_2\}$. Then*

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$

The analyticity of f in the region between the two contours means that one contour can be "deformed" smoothly into the other, without changing the value of the integral. The proof is almost immediate from Cauchy's theorem 5.10:

Proof (of theorem 5.11). Introduce two adjacent straight-line contours: γ_{12}, joining γ_1 to γ_2, and $\gamma_{21} = -\gamma_{12}$, joining γ_2 to γ_1. Then the composite contour $\gamma_1 \cup \gamma_{12} \cup (-\gamma_2) \cup \gamma_{21}$ is a simple closed contour, within which f is analytic. By Cauchy's theorem 5.10 then,

$$\int_{\gamma_1} f(z)\, dz + \int_{\gamma_{12}} f(z)\, dz + \int_{-\gamma_2} f(z)\, dz + \int_{\gamma_{21}} f(z)\, dz = 0.$$

Since $\int_{\gamma_{21}} = \int_{-\gamma_{12}} = -\int_{\gamma_{12}}$ and $\int_{-\gamma_2} = -\int_{\gamma_2}$ the result follows.

We also have the contour deformation theorem for two open contours with the same endpoints:

Theorem 5.12 *Let $f(z)$ be analytic on a domain D, and let a, b be two distinct points in D. Then if γ_1 and γ_2 are any two open contours in D connecting a and b, we have*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Proof If γ_1 and γ_2 do not intersect except at their endpoints then together they form a simple closed curve to which Cauchy's theorem 5.10 applies, and the result is immediate. If γ_1 and γ_2 intersect at one or more points then simply apply Cauchy's theorem to the closed portions of contour between successive pairs of intersection points.

For completeness, we note the other forms of Cauchy's theorem that may be proved:

Theorem 5.13 *(Cauchy's theorem, version 4) If $f(z)$ is analytic in a simply-connected domain D then $\int_\gamma f(z)\,dz = 0$ for every closed path $\gamma \in D$.*

This version is easily proved from 5.10 applied to individual closed loops of γ, inside (and on) each of which f is analytic.

Finally, the theorem may also be proved for a non-simply-connected domain, provided we impose certain conditions on the contour γ: loosely speaking, it must not "wind around" any "holes" in the domain.

We can formalize this concept of "winding number" of a contour γ by the following definition:

Definition 5.14 *(Index of a contour) Let γ be a closed contour and $w_0 \notin \gamma$. The* **index** *(or winding number) of γ about w_0, denoted $n(\gamma; w_0)$, is defined by*

$$n(\gamma; w_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - w_0} dw. \qquad (49)$$

Clearly this definition always gives zero for points w_0 that lie exterior to the closed contour, by Cauchy's theorem 5.10 (because the function $f(z) = 1/(z - w_0)$ is analytic inside and on the contour). However, if w_0 lies inside the contour the integrand has a singularity there, and we expect a nonzero result in general. Consider the simplest case in which $\gamma = C(w_0, r)$, traversed N times. We parametrize γ by $\gamma = \{w(t) : w(t) = w_0 + re^{i\theta},\ 0 \leq \theta \leq 2N\pi\}$ then the formula (49) gives

$$n(\gamma; w_0) = \frac{1}{2\pi i}\int_0^{2N\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = N,$$

so the definition holds true for this case at least. More generally then the result should hold true by adapting the contour deformation results we have already proved – an arbitrary contour wrapping N times round w_0 being equivalent to the circle wrapping N times around it.

The definition 5.14 is given meaning by the following theorems. We state and prove the theorems for $w_0 = 0$, but the results generalize easily to nonzero w_0. Not surprisingly, given its definition and our knowledge of real calculus, the winding number is closely related to the complex logarithm that we came across in §3.5.2. We first prove a result on conditions under which an analytic branch of the logarithm may be found on a domain D, and then use this result to deduce properties of the winding number.

Theorem 5.15 *Let D be a convex domain, and $0 \notin D$. Then there exists a function $f = \log_D$, analytic on D, such that $e^{f(z)} = z\ \forall z \in D$, and*

$$f(z) - f(a) = \int_\gamma \frac{dw}{w}, \tag{50}$$

where γ is any path in D with endpoints a and z. f is uniquely determined up to the addition of an integer multiple of $2\pi i$, and

$$f(z) = \log_D z = \log|z| + i\theta(z)$$

where $\theta(z) \in \arg(z)$ is continuous on D.

Proof As in the proof of theorem 5.5, it is easily checked that with the definition (50) we have $f'(z) = 1/z$ and, since $0 \notin D$ (so that $1/z$ is analytic on D), f is analytic on D. (Note that, without loss of generality, we may in fact take the contour γ joining a and z to be a straight line, since the definition is path-independent – this is easily shown by the deformation theorem 5.12 just proved.)

We also have

$$\frac{d}{dz}(ze^{-f(z)}) = e^{-f(z)} - zf'(z)e^{-f(z)} = 0.$$

Hence, $z = Ce^{f(z)}$ $(C \neq 0)$. Since we seek f such that $e^{f(z)} = z$, $C = 1$.

To check uniqueness, suppose that there exists another function $g(z)$ analytic on D such that $e^{f(z)} = e^{g(z)} = z$ for all $z \in D$. Differentiation of both sides of this relation gives $zf'(z) = zg'(z) \Rightarrow f'(z) = g'(z)$ for all $z \in D$, and thus $f(z) - g(z) = K$, constant, where $e^K = 1$. This gives $K = 2n\pi i$; thus all such representations f differ by additive multiples of $2\pi i$.

The final part of the theorem follows as in our definition of the complex logarithm earlier, using the relation $z = e^{f(z)}$ with $z = |z|e^{i\theta}$ and $f = u + iv$, which gives $|z| = e^u$ and $v = \theta$ (modulo 2π).

Theorem 5.16 *(Properties of the winding number) Let γ be a* **closed** *path with parameter $t \in [a, b]$, and $0 \notin \gamma$. Then*
(1) $n(\gamma; 0)$ is an integer, where $2\pi i n(\gamma; 0) = \int_\gamma w^{-1} dw$
(2) There exists a continuous function $\Theta : [a, b] \to \mathbb{R}$, unique up to an integer multiple of 2π, such that
(i) $\Theta(t) \in \arg(z(t))$ for all $z(t) \in \gamma$;
(ii) $2\pi n(\gamma; 0) = \Theta(b) - \Theta(a)$.

The first part of the theorem relates the winding number to the complex logarithm of the previous theorem 5.15, while the second part guarantees that for such contours γ (not containing the origin) we can ensure that the argument of the complex number $z(t)$ varies in a continuous manner as we traverse the contour. The net change in a continuously-varying argument is directly related to the winding number of the contour.

Proof Let R be a region containing γ, with $0 \notin R$. We cannot yet apply the above theorem 5.15 directly since R may not be convex. Recalling the proof of the covering theorem 5.8, we construct points $a = t_0 < t < 1 < \ldots < t_N = b$ and overlapping balls $B_k = B(z(t_k), \delta)$ $(0 \leq k \leq N)$ whose union contains γ and is contained within R. We write $z_k = z(t_k)$ for the centers of the balls B_k $(0 \leq k \leq N)$, noting that, since γ is closed, $z_N = z(b) = z(a) = z_0$, and the balls B_0 and B_N will coincide. Theorem 5.15 then applies to each B_k, so we can define an analytic logarithm f_k on each B_k, such that

$$f_k(z) - f_k(z_k) = \int_{[z_k, z]} \frac{dw}{w},$$

and

$$f_k(z) = \log|z| + i\theta_k(z), \quad \text{where } \theta_k \in \arg(z),$$

and for $z \in B_k \cap B_{k+1}$ we have $\theta_{k+1}(z) - \theta_k(z) = 2\pi n_k$, where $n_k \in \mathbb{Z}$. In particular, since B_0 and B_N coincide, we may take $f_N(z) = f_0(z)$, so that $\theta_N(z_N) = \theta_0(z_0)$.

Writing $\gamma_k = \{z \in \gamma : z = z(t), t_k \leq t < t_{k+1}\}$ for $0 \leq k \leq N-1$ (the portion of γ between z_k and z_{k+1}; this is contained within B_k according to the Covering theorem 5.8), we then have

$$\begin{aligned}
n(\gamma;0) &= \frac{1}{2\pi i}\int_\gamma \frac{dw}{w} \\
&= \frac{1}{2\pi i}\sum_{k=0}^{N-1}\int_{\gamma_k}\frac{dw}{w} \\
&= \frac{1}{2\pi i}\sum_{k=0}^{N-1}[f_k(z_{k+1}) - f_k(z_k)] \quad \text{by theorem 5.15} \\
&= \frac{1}{2\pi}\sum_{k=0}^{N-1}[\theta_k(z_{k+1}) - \theta_k(z_k)] \quad \text{(real parts cancel)} \\
&= \frac{1}{2\pi}\sum_{k=0}^{N-1}[\theta_{k+1}(z_{k+1}) - 2\pi n_k] - \frac{1}{2\pi}\sum_{k=0}^{N-1}\theta_k(z_k) \\
&= \frac{1}{2\pi}\sum_{k=1}^{N}\theta_k(z_k) - \frac{1}{2\pi}\sum_{k=0}^{N-1}\theta_k(z_k) - \sum_{k=0}^{N-1} n_k \\
&= \frac{1}{2\pi}(\theta_N(z_N) - \theta_0(z_0)) - \sum_{k=0}^{N-1} n_k \\
&= -\sum_{k=1}^{N} n_{k-1} \in \mathbb{Z} \quad \text{(since } \theta_N = \theta_0 \text{ and } z_N = z_0\text{).} \qquad (51)
\end{aligned}$$

The second part of the theorem relates to the fact that the sequence of logarithms f_k defined above do not necessarily give a single continuous function when "patched" together; but if one defines an argument function Θ suitably, then one can ensure continuous variation of argument as the contour γ is followed.

To do this, we use the parametrization of the contour, and first define argument functions on each section of γ: $\Theta_k(t) = \theta_k(z(t))$, $t \in [t_k, t_{k+1}]$. Each $\Theta_k(t)$ is continuous (since $z(t)$ and $\theta_k(z)$ are), but in general there is no continuity between Θ_k and Θ_{k+1} at $t = t_{k+1}$ (they differ by an integer multiple of 2π: $\Theta_{k+1}(t_{k+1}) - \Theta_k(t_{k+1}) = 2\pi n_k$). If we define $\Theta(t)$ recursively by

$$\Theta(t) = \begin{cases} \Theta_0(t) & t \in [t_0, t_1] \\ \Theta_k(t) - (\Theta_k(t_k) - \Theta(t_k)) & t \in (t_k, t_{k+1}],\ 1 \le k \le N-1, \end{cases} \tag{52}$$

then it is easy to check that this definition gives a function Θ continuous on the entire interval, by checking that the right-hand limit at each t_k gives the appropriate value. At $t = t_1$ we have

$$\Theta(t_1^+) = \Theta_1(t_1) - (\Theta_1(t_1) - \Theta(t_1)) = \Theta(t_1),$$

and then at $t = t_k$ ($k > 1$) we have

$$\Theta(t_k^+) = \Theta_k(t_k) - (\Theta_k(t_k) - \Theta(t_k)) = \Theta(t_k).$$

Noting the recursion in the definition of $\Theta(t)$ we can re-express it on the interval $(t_k, t_{k+1}]$ ($k \ge 1$) as

$$\begin{aligned}
\Theta(t) &= \Theta_k(t) - \Theta_k(t_k) + \Theta(t_k) \\
&= \Theta_k(t) - \Theta_k(t_k) + \Theta_{k-1}(t_k) - \Theta_{k-1}(t_{k-1}) + \Theta(t_{k-1}) \\
&= \Theta_k(t) - \Theta_k(t_k) + \Theta_{k-1}(t_k) - \Theta_{k-1}(t_{k-1}) + \Theta_{k-2}(t_{k-1}) - \Theta_{k-2}(t_{k-2}) + \Theta(t_{k-2}) \\
&\vdots \\
&= \Theta_k(t) - (\Theta_k(t_k) - \Theta_{k-1}(t_k)) - (\Theta_{k-1}(t_{k-1}) - \Theta_{k-2}(t_{k-1})) - \\
&\quad \cdots - (\Theta_1(t_1) - \Theta_0(t_1)) \\
&= \Theta_k(t) - \sum_{j=1}^{k} (\Theta_j(t_j) - \Theta_{j-1}(t_j)) \\
&= \Theta_k(t) - 2\pi \sum_{j=1}^{k} n_{j-1}, \quad t \in (t_k, t_{k+1}] \quad (k \ge 1)
\end{aligned} \tag{53}$$

showing explicitly that $\Theta(t)$, since it only ever differs from one of the argument functions by an integer multiple of 2π, is indeed an argument of the complex number z on γ. This proves part 2(i).

Finally, using (53) with $t = t_N = b$ and $k = N - 1$ we have

$$\Theta(b) = \Theta_{N-1}(b) - 2\pi \sum_{j=1}^{N-1} n_{j-1} = \Theta_N(b) - 2\pi \sum_{j=1}^{N} n_{j-1}.$$

Recalling that $\Theta_N(b) = \theta_N(z_N) = \theta_0(z_0) = \Theta_0(a) = \Theta(a)$ (using the equality of the complex logarithms f_N and f_0, and (52)), this leads to

$$\Theta(b) - \Theta(a) = -2\pi \sum_{j=1}^{N} n_{j-1} \equiv 2\pi n(\gamma; 0),$$

proving 2(ii).

An immediate consequence of this theorem is that for closed contours γ that satisfy the conditions of the theorem, $n(\gamma; 0)$ does indeed give the number of times that γ encircles the origin. This is because the function $\Theta(t)$ defines a continuously-varying argument of the complex number $z(t)$ as it moves along γ; thus $\Theta(b) - \Theta(a)$ is the net change in the argument of a point z as it completes a full circuit of γ. Since each complete circuit of the origin corresponds to a change of 2π in the argument of z, a change of $2\pi n(\gamma; 0)$ corresponds to $n(\gamma; 0)$ complete circuits of the origin.

Theorem 5.17 *(Cauchy's theorem – FINAL version! Let f be analytic on an open region D. If γ is a closed contour in D such that $n(\gamma; w) = 0$ $\forall w \notin D$, then $\int_\gamma f(z)\, dz = 0$.*

6 Beyond Cauchy's (first) theorem

Cauchy's theorem, as we are beginning to see, is a very powerful result that forms one of the basic building-blocks of complex analysis. In this section we will continue this process of building, working towards ever-stronger results. We will see, for instance, that analyticity of a function implies that it is **infinitely differentiable**, a result that is certainly not true for (say) real-valued functions that are continuously differentiable.

We begin by proving another result due to Cauchy that gives us an integral representation of any analytic function. It is this key result that will lead to the conclusion of infinite smoothness.

Theorem 6.1 *(Cauchy's Integral Formula) Let $f(z)$ be analytic inside and on a simple closed contour γ. Then for any z inside γ ($z \in I(\gamma)$),*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w-z} dw. \tag{54}$$

This theorem is remarkable – it says that, if a function is known to be analytic inside and on a given curve, then its values on that curve are sufficient to determine uniquely its values everywhere inside the curve. Recall the connection that we know exists between harmonic functions and analytic functions – we know from PDE theory (or we should know!) that, if we wish to solve Laplace's equation in a given domain, then the solution u is uniquely determined by the values of u at all points of the boundary. We also know (lemma 3.24) that a harmonic conjugate v of u may be defined, unique up to an additive constant, such that $u + iv$ is a complex analytic function. Hence theorem 6.1 does fit with several facts we already knew. What we **cannot** do, however, is **arbitrarily prescribe** the values of a function f on a curve γ and expect (54) to yield an analytic function at all points of the interior of γ. As the results for Laplace's equation suggest, in fact we can really only arbitrarily prescribe either the real or imaginary part of an analytic function on a simple closed contour γ – and this is sufficient to uniquely determine the values of the corresponding analytic function at all points within γ (at least, up to an arbitrary constant). The real and imaginary parts of f are closely related, and cannot be prescribed independently, even on a contour in $\mathbb{C}$.

Proof Let $z \in I(\gamma)$, then since $I(\gamma)$ is open $\exists \epsilon > 0$ such that $B(z;\epsilon) \subset I(\gamma)$. For any $0 < \delta < \epsilon$,

$$\oint_\gamma \frac{f(z)}{w-z} dw = \int_{C(z;\delta)} \frac{f(z)}{w-z} dw, \quad \text{and} \quad \oint_\gamma \frac{f(w)}{w-z} dw = \int_{C(z;\delta)} \frac{f(w)}{w-z} dw,$$

(where $C(z;\delta)$ denotes the circle with center z and radius δ), by the deformation theorem 5.11. Thus, using the result

$$\oint_{C(z;\delta)} \frac{dw}{w-z} = \int_0^{2\pi} \frac{i\delta e^{i\theta}\, d\theta}{\delta e^{i\theta}} = 2\pi i,$$

in which we parametrized $C(z;\delta)$ by $w = z + \delta e^{i\theta}$, $0 \leq \theta < 2\pi$, we have

$$\left| \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w-z} dw - f(z) \right| \quad = \quad \left| \frac{1}{2\pi i} \oint_{C(z;\delta)} \frac{f(w) - f(z)}{w-z} dw \right|$$

$$
\begin{aligned}
&= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+\delta e^{i\theta}) - f(z)}{\delta e^{i\theta}} i\delta e^{i\theta} d\theta \right| \\
&\leq \frac{1}{2\pi} \sup_{0 \leq \theta < 2\pi} |f(z+\delta e^{i\theta}) - f(z)|.
\end{aligned}
$$

Since f is analytic at z it is certainly continuous (Lemma 3.16), hence the last expression on the right-hand-side goes to zero as $\delta \to 0$. Since $\delta > 0$ was arbitrary, the theorem is proved.

Corollary 6.2 *(Cauchy's formula for derivatives) If f is analytic inside and on a simple closed contour γ then all its derivatives exist in $I(\gamma)$, and*

$$
f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^{k+1}} dw. \tag{55}
$$

Proof For $k = 0$ this is just theorem 6.1 proved above. So, we suppose (55) holds for $k = n$, and show that it must then hold for $k = n + 1$ (the general result will then hold, by induction). As above we may use theorem 5.11 to deform the contour of integration to a (sufficiently small) circle, $C = C(z; 2r)$ for some fixed $r > 0$. We use the definition of derivative, (23), to try to evaluate $f^{(n+1)}(z)$ from $f^{(n)}(z)$ which, by our induction hypothesis, is given by (55).

We now let $h \in \mathbb{C}$ with $|h| < r$. Then

$$
\begin{aligned}
f^{(n)}(z+h) - f^{(n)}(z) &= \frac{n!}{2\pi i} \oint_C f(w) \left(\frac{1}{(w-z-h)^{n+1}} - \frac{1}{(w-z)^{n+1}} \right) dw \\
&= \frac{(n+1)!}{2\pi i} \oint_C f(w) \left(\int_{[z,z+h]} \frac{1}{(w-\zeta)^{n+2}} d\zeta \right) dw,
\end{aligned}
$$

using the Fundamental Theorem of Calculus 4.4. Then

$$
\begin{aligned}
F(h) &:= \frac{f^{(n)}(z+h) - f^{(n)}(z)}{h} - \frac{(n+1)!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+2}} dw \\
&= \frac{(n+1)!}{2\pi i h} \oint_C f(w) \left(\int_{[z,z+h]} \frac{1}{(w-\zeta)^{n+2}} - \frac{1}{(w-z)^{n+2}} d\zeta \right) dw \\
&= \frac{(n+2)!}{2\pi i h} \oint_C f(w) \left(\int_{[z,z+h]} \left(\int_{[z,\zeta]} \frac{1}{(w-\tau)^{n+3}} d\tau \right) d\zeta \right) dw,
\end{aligned}
$$

again applying theorem 4.4. Since f is analytic, it is continuous, and therefore bounded (by M, say) on the compact set C. Note that for any $w \in C = C(z; 2r)$, we have $|w - z| = 2r$, while $|h| < r$; and in the above integrals,

$$\tau \in [z, \zeta], \quad \zeta \in [z, z+h] \quad \Rightarrow \quad |z - \zeta| \leq |h| < r \quad \text{and} \quad |w - \tau| \geq r.$$

Using the estimation theorem 4.7 then:

$$|F(h)| \leq \frac{M(n+2)!}{2\pi|h|} \times 4\pi r \times |h| \times |h| \times \frac{1}{r^{n+3}} \to 0 \quad \text{as } |h| \to 0,$$

for fixed $r > 0$. The result is therefore proved.

As well as being useful in proving many important subsequent results, theorems 6.1 and 6.2 can also provide us with a quick way to evaluate certain complex integrals.

Example 6.3 *Evaluate the integrals*

$$(a) \int_\gamma \frac{w^3+5}{w-i} dw, \quad (b) \int_\gamma \frac{1}{w^2+w+1} dw, \quad (c) \int_\gamma \frac{\sin w}{(w+4)^2(w-1)^3} dw.$$

where γ is the circle $C(0; 2)$ (with center 0 and radius 2).

For (a) we use the result (54) with $f(w) = w^3 + 5$ and $z = i$ to deduce that

$$\int_\gamma \frac{w^3+5}{w-i} dw = 2\pi i f(i) = 2\pi i(5 - i).$$

For (b) we note that $w^2+w+1 = (w-w_0)(w-\overline{w}_0)$, where $w_0 = (-1+i\sqrt{3})/2$. Since $|w_0| = 1$ both zeros of the denominator lie within the integration contour, and we cannot apply (54) directly (whether we choose $f(w) = 1/(w - w_0)$ and $z = \overline{w}_0$, or $f(w) = 1/(w + \overline{w}_0)$ and $z = w_0$, we always have a singularity of f inside the integration contour). However, we can use partial fractions to decompose the integrand as

$$\int_\gamma \frac{1}{w^2+w+1} dw = \frac{1}{w_0 - \overline{w}_0} \int_\gamma \left(\frac{1}{w-w_0} - \frac{1}{w-\overline{w}_0} \right) dw = \frac{1}{w_0 - \overline{w}_0}(1-1) = 0,$$

applying (54) to each term of the decomposition, with $f(w) = 1$ and $z = w_0$, $z = \overline{w}_0$, respectively.

For (c) we use (55) with $k = 2$, $f(w) = \sin w/(w+4)^2$ and $z = 1$ to obtain

$$\int_\gamma \frac{\sin w}{(w+4)^2(w-1)^3} dw = \frac{2\pi i}{2!} f''(1).$$

The integrals in this example could all be worked out by a similar, but much more general method, using Cauchy's residue theorem, which we shall soon derive. (Both Cauchy's integral formula and Cauchy's formula for derivatives may be thought of as special cases of the Residue theorem).

Corollary 6.4 *(Mean value representations for f) If f is analytic inside and on a circular contour of radius R centered at z then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\theta}) d\theta \tag{56}$$

and

$$f(z) = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} f(z + re^{i\theta}) r dr d\theta. \tag{57}$$

These representations (56) and (57) state that the value of f at the point z is equal to its mean value integrated around the circle of radius R, and is also equal to its mean value integrated over the area of the same circle, respectively. [Of course, Cauchy's integral formula (54) itself may be considered as a generalized mean value result relating the value of f at any point $z \in \gamma$ to a weighted average of its values along γ.] Proofs of (56) and (57) are easy – for (56) just take the contour γ in theorem 6.1 to be $\gamma = \{z \in \mathbb{C} : z = Re^{i\theta}, 0 \leq \theta < 2\pi\}$. Then (57) follows by multiplying both sides of (56) by $r\,dr$ and integrating from $r = 0$ to $r = R$.

Theorem 6.5 *(Maximum modulus theorem, 1) (i) If f is analytic in a domain D then $|f|$ cannot have a maximum in D unless f is constant. (ii) If f is analytic in a bounded domain D and $|f|$ is continuous on the closed domain $\overline{D}$, then $|f|$ assumes its maximum on the boundary of the domain.*

Proof We use (57) to establish this theorem, by contradiction. Suppose that $|f|$ has an interior maximum at $z \in D$, $|f(z)| \geq |f(w)|\ \forall w \in D$. Since z is an interior point we can fit a ball of some radius R around it such that

$B(z;R) \subset D$. Writing $\zeta = z + re^{i\theta}$ for any point ζ in this ball $(0 \leq r < R)$ (57) then gives

$$\begin{aligned} f(z) &= \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} f(\zeta) r dr d\theta \\ \Rightarrow \quad |f(z)| &= \frac{1}{\pi R^2} \left| \int_{B(z;R)} f(\zeta) dS \right| \\ &\leq \frac{1}{\pi R^2} \int_{B(z;R)} |f(\zeta)| dS \\ &\leq \frac{|f(z)|}{\pi R^2} \int_{B(z;R)} dS \\ &= |f(z)|, \end{aligned}$$

where we used a variant of the estimation theorem 4.7 (for double integrals; the proof is exactly similar) in the first inequality, and the contradiction assumption in the second. Since we end up again with $|f(z)|$ at the end of this chain of inequalities we must have equality throughout; and in particular,

$$|f(z)| = \frac{1}{\pi R^2} \int_{B(z;R)} |f(\zeta)| dS$$
$$\Rightarrow \quad \frac{1}{\pi R^2} \int_{B(z;R)} (|f(z)| - |f(\zeta)|) dS = 0.$$

The integrand here is real and non-negative; so the only way it can integrate to zero is if it is identically zero. This implies that $|f(\zeta)| = |f(z)|$ on $B(z;R)$, and hence f is constant on this ball. (The fact that f is constant on a region where $|f|$ is constant is a simple deduction from the Cauchy-Riemann equations.)

The above shows that f must in fact be constant on the largest ball that can fit around an interior maximum point z within D. Since all points of this ball are now maxima of $|f|$ we can repeat the argument for a different point near the ball boundary and, proceeding in this way, cover the whole of D with balls on which f must be constant. This proves part (i).

The proof of (ii) is almost immediate from (i). The continuity of the real function $|f|$ on the closed bounded region $\overline{D}$ (a compact set) guarantees that $|f|$ is bounded and attains its bounds; so we know that $|f|$ does have a maximum on $\overline{D}$. Part (i) tells us that there are then two possibilities: if $|f|$ has a maximum in D, then f is constant on D and, by continuity therefore,

f is constant on $\overline{D}$, so that $|f|$ attains its maximum on ∂D trivially. The second possibility is that $|f|$ is non-constant, and has no maximum in D. Then the only possibility is that the maximum of $|f|$ on $\overline{D}$ occurs on ∂D.

6.1 Morera and Liouville theorems

We have already seen that non-analytic functions can integrate to zero around simple closed contours in the complex plane – recall example 4.3 earlier – so certainly the condition that a given function f integrate to zero around a given contour is not sufficient to guarantee analyticity of f. Morera's theorem gives the necessary converse to Cauchy's theorem:

Theorem 6.6 *(Morera's theorem – a Cauchy converse) If $f(z)$ is continuous in a domain D and if*

$$\oint_\gamma f(z)\,dz = 0$$

for every simple closed contour γ lying in D, then $f(z)$ is analytic in D.

Proof Choose an arbitrary point $\zeta \in D$. Since D is open, $\exists \delta > 0$ such that $B(\zeta;\delta) \subset D$. Define a function $F(z)$ on this ball by

$$F(z) = \int_{[\zeta,z]} f(w)\,dw$$

(remember theorem 5.5). Then, exactly as shown in the proof of theorem 5.5, F is analytic on the ball, with $F' = f$. From the theorem 6.2 (Cauchy's formula for derivatives), in fact F is infinitely differentiable; thus f is too. Hence f is analytic on $B(\zeta;\delta)$. Since the point $\zeta \in D$ was arbitrary, we conclude that f is analytic on D.

Remark In fact a slightly stronger (though not much more useful) version of Morera's theorem could be stated above: we only require that f integrate to zero around any given **triangular** contour in D (the construction used to demonstrate that $F' = f$ uses only this fact).

Definition 6.7 *(Entire function) A function that is analytic in the whole complex plane $\mathbb{C}$ is called an entire function.*

Entire functions are quite rare in the space of all complex functions, and strong results can be proved for them. There is some ambiguity in the literature regarding the notion of analyticity of a function at infinity. Some would define a function $f(z)$ to be analytic at infinity only if $\tilde{f}(z) := f(1/z)$ is analytic at the origin. However, it is normally accepted that functions such as complex polynomials (e.g. z^n) – which do not satisfy the above criterion for analyticity at infinity, are entire.

Theorem 6.8 *(Liouville) If $f(z)$ is entire and bounded in $\mathbb{C}$ (including at infinity) then $f(z)$ is constant.*

Proof The proof uses the expression (55) for $f'(z)$. Since f is entire we may choose any contour γ we wish in this expression, and we choose the circle of arbitrary radius R centered on z: $|\zeta - z| = R$, or $\zeta = z + Re^{i\theta}$, $0 \leq \theta < 2\pi$. Then, since we know $|f| \leq M$ for some $M > 0$, and $d\zeta = iRe^{i\theta}d\theta$, we have

$$|f'(z)| = \frac{1}{2\pi}\left|\oint_C \frac{f(\zeta)}{(\zeta - z)^2}d\zeta\right| \leq \frac{1}{2\pi}\oint_C \frac{|f(\zeta)|}{|\zeta - z|^2}|d\zeta| \leq \frac{M}{2\pi R}\int_0^{2\pi} d\theta = \frac{M}{R}.$$

Since the contour radius R is arbitrary, we may take it to be as large as we wish, and we conclude that $f'(z) = 0$, so that f is constant as claimed.

We note without proof the following much stronger result due to Picard:

Theorem 6.9 *(Picard's little theorem) If f is entire and non-constant, then the range of f is either the entire complex plane, or the plane minus a single point.*

Homework: Ablowitz & Fokas, problems for section 2.6. Questions 1 (a),(b),(c),(d) (integrate by direct parametrization), 3,4,5,7. You could also try 9.

Liouville's theorem can be used to give a quick and elegant proof of the fundamental theorem of algebra:

Theorem 6.10 *(Fundamental theorem of algebra) Let $p(z)$ be a non-constant polynomial, with complex coefficients. Then $\exists \zeta \in \mathbb{C}$ with $p(\zeta) = 0$.*

Proof The proof proceeds by contradiction: we suppose that $p(z) \neq 0 \; \forall z \in \mathbb{C}$. Since $|p(z)| \to \infty$ as $|z| \to \infty$, $\exists\ R > 0$ such that $1/|p(z)| < 1$ for $|z| > R$. On $|z| \le R$, which is a compact set, $1/|p(z)|$ is continuous, and hence bounded. Hence $1/p(z)$ is bounded on the whole complex plane. It is also analytic (since $p(z)$ nowhere vanishes; see homework question (2) in §3.3). By Liouville's theorem 6.8 then, $p(z)$ must be constant. This contradiction proves the theorem.

It is a simple matter to prove inductively from this result that a polynomial of order N (such that $|p(z)| = O(|z|^N)$ as $|z| \to \infty$) has exactly N zeros, counted according to multiplicity.

Another result that may be deduced from Liouville's theorem relates to the behavior of entire functions at infinity:

Theorem 6.11 *Suppose $f(z)$ is an entire function such that $\lim_{|z|\to\infty} |f(z)/z^N| = M$, $0 < M < \infty$. Then $f(z)$ is a polynomial of degree N.*

Remark The condition $\lim_{|z|\to\infty} |f(z)/z^N| = M$, $0 < M < \infty$, is often written $|f(z)/z^N| = O(1)$ as $|z| \to \infty$, or equivalently, $|f(z)| = O(|z|^N)$ as $|z| \to \infty$.

7 Taylor series and related results

7.1 Taylor's theorem

Given that we now know analytic functions to be infinitely differentiable within their domain of analyticity, we might expect to be able to express such functions in terms of convergent power series, as we can real functions. This turns out to be the case (in fact the result is stronger in the complex than the real case), and the following theorem gives the exact result.

Theorem 7.1 *(Taylor series) Suppose $f(z)$ is analytic on $B(z_0; R)$. Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z - z_0)^n}{n!}. \tag{58}$$

This is the Taylor series expansion of f about the point z_0, and it converges uniformly on $|z - z_0| \le R$.

The theorem utilizes Cauchy's integral formula, and in the proof we will need to interchange the order of summation and integration. This is always fine for a finite sum, but can pose problems when the sum is infinite, as in the theorem statement. We shall therefore first prove the following theorem (and a corollary):

Theorem 7.2 *Let G_n be a sequence of continuous functions, with $G_n(z) \to G(z)$ as $n \to \infty$, uniformly on a region D. Then $G(z)$ is continuous, and for any finite contour $\gamma \in D$,*

$$\lim_{n\to\infty} \int_\gamma G_n(z)\, dz = \int_\gamma \lim_{n\to\infty} G_n(z)\, dz = \int_\gamma G(z)\, dz. \tag{59}$$

By uniform convergence on D here we mean that, for any $z \in D$, and any $\epsilon > 0$, $\exists\ N$, dependent on ϵ but **independent** of z, such that $|G_n(z) - G(z)| < \epsilon$ for all $n \geq N$.

Proof We first prove continuity of G. For any two points z, z_0 in D we have

$$G(z) - G(z_0) = G_n(z) - G_n(z_0) + G_n(z_0) - G(z_0) + G(z) - G_n(z)$$

so that

$$|G(z) - G(z_0)| \leq |G_n(z) - G_n(z_0)| + |G_n(z_0) - G(z_0)| + |G(z) - G_n(z)|.$$

Uniform convergence of the sequence on D means that given $\epsilon_1 > 0$, $\exists N$ (independent of z or z_0, but depending on ϵ_1) such that

$$|G_n(z_0) - G(z_0)| < \epsilon_1/3 \quad \text{and} \quad |G_n(z) - G(z)| < \epsilon_1/3, \quad \forall n \geq N.$$

Since all the G_n are continuous on D $\exists \delta > 0$ such that

$$|G_n(z) - G_n(z_0)| < \epsilon_1/3 \quad \text{for } |z - z_0| < \delta.$$

Thus, for $n \geq N$ and $|z - z_0| < \delta$, we have

$$|G(z) - G(z_0)| < \epsilon_1,$$

so G is continuous on D. It remains to prove that we can take the limit of the sequence through the integral. Let $\epsilon > 0$ be arbitrary. By uniform convergence of the sequence G_n to G we know we can find N such that, for $n \geq N$, $|G_n(z) - G(z)| < \epsilon/L(\gamma)$. Then,

$$\left|\int_\gamma G_n(z)\,dz - \int_\gamma G(z)\,dz\right| \leq \int_\gamma |G_n(z) - G(z)||dz| < \frac{\epsilon}{L(\gamma)}L(\gamma) = \epsilon,$$

using theorem (4.8). The result follows.

Corollary 7.3 *(Integrals of convergent series) If $g_n(z)$ is a sequence of continuous functions on D such that the series $\sum_n g_n(z)$ converges uniformly to the sum $G(z)$ on D, then for any closed curve $\gamma \in D$,*

$$\sum_{n=1}^{\infty}\left(\int_\gamma g_n(z)\,dz\right) = \int_\gamma\left(\sum_{n=1}^{\infty} g_n(z)\,dz\right) = \int_\gamma G(z)\,dz.$$

Proof Apply theorem 7.2 to the sequence of partial sums $G_n(z) = \sum_{k=0}^{n} g_k(z)$, using the fact that each $G_n(z)$ is continuous on D and $G_n(z) \to G(z)$ uniformly as $n \to \infty$.

Proof (of Taylor's theorem 7.1) We use Cauchy's integral formula. Let $z_0 \in B(z_0; R)$ and choose $r > 0$ such that $|z - z_0| < r < R$. Let $\gamma = C(z_0; r)$, then (54) gives

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z)}. \tag{60}$$

We write the factor $1/(w-z)$ here as

$$\frac{1}{w-z} = \frac{1}{(w-z_0)}\frac{1}{(1-[(z-z_0)/(w-z_0)])}$$

and, noting that for $w \in \gamma$ we have $|z - z_0| < |w - z_0|$, we can expand the right-hand side here as a convergent series. Thus, (60) becomes

$$f(z) = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} f(w)\,dw. \tag{61}$$

The contour γ is a compact set, on which the analytic (therefore continuous) function f is bounded, by M say. Thus,

$$\left|\frac{(z-z_0)^n}{(w-z_0)^{n+1}} f(w)\right| \leq \frac{M}{r}\left(\frac{|z-z_0|}{r}\right)^n =: M_n \quad \text{(say)},$$

where $\sum M_n$ converges, since $|z - z_0| < r$. By theorem 7.2 and its corollary 7.3, summation and integration may be interchanged in (61), since the integrand is uniformly convergent on γ, giving

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}}\, dw \right) (z - z_0)^n.$$

From Cauchy's formula for derivatives (55), we note that this is exactly

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

and the theorem is proved.

Strictly speaking, the proof above shows only that the expansion (58) is one possible power series expansion of a given analytic function f about the point z_0. Unsurprisingly, however, the power series expansion of f about z_0 is unique, as we will shortly prove.

Example 7.4 *Find the Taylor series expansion of (a) $f(z) = e^z$ about $z = a$, (b) $f(z) = 1/z$ about $z = 1$.*

For (a) we note that $f^{(n)}(z) = e^z$, $\forall n$, and so we have the Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{e^a (z - a)^n}{n!}. \tag{62}$$

This power series is convergent for all $z \in \mathbb{C}$ (infinite radius of convergence). Note that if $a = 0$ we get exactly the usual exponential power series, as we would expect.

For (b) we have

$$f^{(n)}(z) = (-1)^n n! z^{-(n+1)},$$

and thus the Taylor series about $z = 1$ is given by

$$f(z) = \sum_{n=0}^{\infty} (-1)^n (z - 1)^n, \tag{63}$$

convergent for $|z-1| < 1$. Here we note that the function f can be rewritten as

$$f(z) = z^{-1} = (1 + (z-1))^{-1}$$

which, for $|z-1| < 1$ can be expanded binomially to give exactly the result (63). We will show soon that the power series expansion for an analytic function about a given point is unique, so that if, by *any* means, we are able to find a power series expansion of the function about the given point, then that expansion must be its Taylor series. The above example (b) illustrates this fact. Alternatively, once we know that the Taylor expansion is unique, we can view example (b) as a proof that the binomial theorem is valid here. Similarly for example (a) there are alternative ways to obtain the power series – we could write $\zeta = z - a$ and $e^z = e^{a+\zeta} = e^a e^\zeta$. Then use the definition of e^ζ, to obtain the same result as (62).

Example 7.5 *(Binomial expansion) Show that, for positive integers n, the binomial expansion*

$$(z+h)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} h^{n-k} z^k$$

is valid for all $z \in \mathbb{C}$.

Writing $f(z) = (z+h)^n$, $f(z)$ is analytic for all $z \in \mathbb{C}$. Thus Taylor's theorem is applicable for all $z \in \mathbb{C}$.

$$f^{(k)}(z) = n(n-1)\ldots(n-k+1)(z+h)^{n-k} = \frac{n!}{(n-k)!}(z+h)^{n-k}, \quad k \le n$$

$(f^{(n)}(z) = n!)$, and

$$f^{(k)}(z) = 0, \quad k > n.$$

By Taylor's theorem then,

$$(z+h)^n = \sum_{k=0}^{n} \frac{f^{(k)}(0) z^k}{k!} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} h^{n-k} z^k \equiv \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} z^{n-k} h^k.$$

This example shows that the "standard" binomial expansion holds good for complex as well as real variables.

The power series expansion in Taylor's theorem 7.1 contains only positive powers of $(z - z_0)$, reflecting the fact that the function f is analytic at the point z_0 (so the power series is well-behaved there). If we know only that f is analytic in some **annulus** about z_0, $R_1 \leq |z - z_0| \leq R_2$, then we will see that we can find a more general **Laurent series** expansion for $f(z)$ on the annulus, that contains both positive and negative powers of $(z - z_0)$. Clearly any function whose power series expansion about z_0 contains negative powers of $(z-z_0)$ is **singular** at the point z_0. We discuss singularities in detail below, but first we note a couple of useful consequences of Taylor's theorem 7.1.

The proof above demonstrates the uniform convergence of the Taylor series about z_0 for balls on which f is analytic. Therefore we are able to differentiate the Taylor series term-by-term to obtain the Taylor series of the derivative:

Theorem 7.6 *(Termwise differentiation) If $f(z)$ is analytic for $|z-z_0| < R$, given by the Taylor series (58), then the series obtained by differentiating the Taylor series term-by-term converges uniformly to f' in $|z - z_0| < R$.*

Proof We know (corollary 6.2) that f analytic on $|z - z_0| < R$ implies analyticity of f' on $|z - z_0| < R$ (and of all higher derivatives of f). Thus Taylor's theorem 7.1 applies to f', giving its Taylor expansion as

$$f'(z) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(z_0)}{n!}(z - z_0)^n, \tag{64}$$

uniformly convergent for $|z - z_0| < R$ by the arguments in the proof of 7.1. On the other hand, termwise differentiation of the Taylor series for f about z_0 gives

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n\right) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{(n-1)!}(z - z_0)^{n-1} = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(z_0)}{n!}(z - z_0)^n.$$

Hence the term-by-term differentiation of f gives the same result as direct application of Taylor's theorem to f', as claimed.

Taylor's theorem gives us a recipe for finding convergent power series expansions of given analytic functions. It is also useful to be able to deduce properties of a given power series. The following theorem tells us that, where a given power series in z is uniformly convergent, it necessarily defines an analytic function; and within its region of uniform convergence it may (like a Taylor series) be differentiated term-by-term to obtain the function's derivative.

Theorem 7.7 *If the power series defined by*

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \tag{65}$$

converges uniformly for $|z - z_0| < R$ then it is analytic on this disk, and can be differentiated termwise to obtain a uniformly convergent series for $|z - z_0| < R$.

Proof For the proof we take $z_0 = 0$ for simplicity, noting that the result is easily extended to general z_0 by a shift of origin. Let $g(z)$ be the power series defined by

$$g(z) = \sum_{n=1}^{\infty} n b_n z^{n-1} = \sum_{n=1}^{\infty} g_n(z). \tag{66}$$

Then $g(z)$ has the same radius of convergence as $f(z)$, because if we take $z \in B(0; R)$ and $\rho \in (|z|, R)$, then

$$|g_n(z)| = |n b_n z^{n-1}| = \frac{n}{|z|} \left(\frac{|z|}{\rho} \right)^n |b_n \rho^n|.$$

The series $\sum n(|z|/\rho)^n$ converges for $|z| < \rho$, by the ratio test for real series. Therefore its terms (which are all positive) are all bounded, $n(|z|/\rho)^n \leq M$ say, for some $M > 0$. Then

$$|g_n(z)| \leq \frac{M}{|z|} |b_n \rho^n|,$$

and so $\sum g_n(z)$ is absolutely convergent for $0 < |z| < R$, by the comparison test. The absolute convergence of the series $\sum_n g_n(z)$ for $z = 0$ is immediate from setting $z = 0$ in the sum; thus we have absolute convergence for all $|z| < R$. We note that the same argument applied to the power series $\sum_n g_n(z)$ also guarantees absolute convergence of $\sum_{n=2}^{\infty} n(n-1) b_n z^{n-2}$ on $B(0; R)$, a result we shall use below.

We must now show the analyticity of the general power series (65) within its radius of convergence: we shall show that $f'(z)$ exists and is given by $g(z)$. Let z, $z+h \in B(0;R)$, then

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| = \left|\sum_{n=1}^{\infty} b_n \left(\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right)\right|$$

(the term in $n=0$ is identically zero). By the binomial expansion (example 7.5), the term $(z+h)^n$ on the right-hand side here can be written

$$(z+h)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k,$$

giving

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h} - g(z) &= \frac{1}{h}\sum_{n=1}^{\infty} b_n \left(\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} h^k z^{n-k} - z^n\right) - \sum_{n=1}^{\infty} b_n n z^{n-1} \\
&= \sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} \frac{n!}{(n-k)!k!} h^{k-1} z^{n-k} - \sum_{n=1}^{\infty} b_n n z^{n-1} \\
&= \sum_{n=1}^{\infty} b_n \sum_{k=2}^{n} \frac{n!}{(n-k)!k!} h^{k-1} z^{n-k} \\
&= h \sum_{n=1}^{\infty} b_n \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left|\frac{f(z+h)-f(z)}{h} - g(z)\right| &= |h| \left|\sum_{n=1}^{\infty} b_n \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-2}\right| \\
&\leq |h| \sum_{n=1}^{\infty} \frac{n(n-1)}{2} |b_n| \sum_{m=0}^{n-2} \binom{n-2}{m} |z|^{n-2-m} |h|^m \\
&\leq |h| \sum_{n=1}^{\infty} \frac{n(n-1)}{2} |b_n| (|z|+|h|)^{n-2}, \qquad (67)
\end{aligned}$$

where we used the result

$$\begin{aligned}
\binom{n}{k} &= \frac{n(n-1)(n-2)!}{k(k-1)(n-k)!(k-2)!} = \frac{n(n-1)(n-2)!}{(m+2)(m+1)(n-2-m)!m!} \\
&\leq \frac{n(n-1)}{2} \binom{n-2}{m}
\end{aligned}$$

with $m = k - 2 \geq 0$. Choosing $\rho \in (|z|, R)$, we now recall that, as shown above, $\sum_{n=2}^{\infty} n(n-1)|b_n|\rho^{n-2}$ converges (to K, say). It follows that for $|h| < \rho - |z|$,

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| \leq \frac{|h|}{2} \sum_{n=1}^{\infty} n(n-1)|b_n|\rho^{n-2} \leq \frac{K|h|}{2}.$$

Hence $f'(z)$ exists and equals $g(z)$, as claimed.

Note The above proof can be easily adapted to the power series (65) for arbitrary $z_0 \neq 0$.

Corollary 7.8 *The power series defined by (65), with radius of convergence $R > 0$, has derivatives of all orders within $B(z_0; R)$. In particular, $f^{(n)}(z_0) = n!b_n$.*

It is immediate from this corollary that the coefficients in a power series expansion are unique. In particular, the Taylor series expansion of theorem 7.1 is unique.

7.1.1 Taylor expansion about infinity

The foregoing discussion implicitly assumed that we are dealing with the Taylor series expansion of a function about some finite point $a \in \mathbb{C}$, but the ideas are applicable to the point at infinity also. To obtain the Taylor series of a function about infinity, we simply write $\zeta = 1/z$ and the Taylor expansion about $\zeta = 0$ then provides the appropriate Taylor series at infinity.

Example 7.9 *Find the Taylor series expansions of (a) $f(z) = e^{1/z}$ and (b) $f(z) = (az - b)/(cz - d)$ about the point $z = \infty$.*

For (a), note that $f(\zeta) = e^{\zeta} = \sum_{n=0}^{\infty} \zeta^n/n!$ gives the Taylor series about $\zeta = 0$ (infinite radius of convergence). Therefore the Taylor series of f about $z = \infty$ is

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n n!}.$$

For (b), writing $\zeta = 1/z$ gives

$$\begin{aligned} f(\zeta) &= \frac{a-b\zeta}{c-d\zeta} = c(a-b\zeta)(1-\frac{d\zeta}{c})^{-1} = c(a-b\zeta)\sum_{n=0}^{\infty}\left(\frac{d\zeta}{c}\right)^n \\ &= ac\sum_{n=0}^{\infty}\left(\frac{d}{c}\right)^n\zeta^n - bc\sum_{n=0}^{\infty}\left(\frac{d}{c}\right)^n\zeta^{n+1} \\ &= ac + \sum_{n=1}^{\infty}\left(\frac{d}{c}\right)^{n-1}(ad-bc)\zeta^n. \end{aligned}$$

This power series converges absolutely for $|\zeta| < |c/d|$. Thus the Taylor series of the function about infinity is

$$f(z) = ac + \sum_{n=1}^{\infty}\left(\frac{d}{c}\right)^{n-1}(ad-bc)z^{-n},$$

convergent for $|z| > |d/c|$.

Example 7.10 *Can we find the Taylor expansion of the function e^z about $z = \infty$?*

Again we write $\zeta = 1/z$, then the power series expansion of $e^{1/\zeta}$ about $\zeta = 0$ is what we need. However, $e^{1/\zeta}$ is not analytic in any neighborhood of $\zeta = 0$! Taylor's theorem is not applicable, and there is no Taylor series expansion of e^z at $z = \infty$ – the function is not analytic there.

Multiplication of power series

We conclude this section by noting that two power series may be multiplied together within the radius of convergence of both, to get a single uniformly convergent power series. This result is familiar from real analysis. We can justify the procedure here by noting that, if we have two power series that are uniformly convergent on some common domain $|z| < R$, then each of these series defines a function analytic on $|z| < R$, by theorem 7.7:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad g(z) = \sum_{n=0}^{\infty} g_n z^n, \quad f,\, g \text{ analytic on } |z| < R.$$

The product of these functions, $f(z)g(z)$, is then also analytic on $|z| \leq R$ (the product of two analytic functions is another analytic function – Exercise:

prove this from first principles) which, by Taylor's theorem 7.1, has a power series expansion

$$f(z)g(z) = \sum_{n=0}^{\infty} (fg)_n z^n,$$

absolutely convergent on $|z| < R$, where the coefficients $(fg)_n$ are given by the usual formula in theorem 7.1. We then have

$$\sum_{n=0}^{\infty} (fg)_n z^n = \left(\sum_{n=0}^{\infty} f_n z^n \right) \left(\sum_{n=0}^{\infty} g_n z^n \right), \quad |z| < R.$$

Comparison of terms on left- and right-hand sides here then gives the usual formula

$$(fg)_n = \sum_{r=0}^{n} f_r g_{n-r}$$

for the coefficient of z^n, using the fact that the power series (Taylor series) expansion of an analytic function is unique.

7.2 Zeros of analytic functions

It is often important to know where a given analytic function vanishes on its domain of analyticity, as this can have implications for singularities of related functions, and knowledge of zeros can even sometimes tell us that a function vanishes identically on a domain (see the Identity theorem 7.13 below). Taylor's theorem gives a straightforward way of characterizing zeros of an analytic function f.

Definition 7.11 *(Order of a zero) Suppose f is analytic at $a \in \mathbb{C}$, and $f(a) = 0$. We say that a is a zero of order m if*

$$0 = f(a) = f'(a) = \ldots = f^{(m-1)}(a) \quad \textit{and} \quad f^{(m)}(a) \neq 0.$$

A zero of order 1 is usually called a simple zero.

This is a sensible definition since the Taylor series for such a function about $z = a$ takes the form

$$f(z) = \sum_{n=m}^{\infty} c_n (z-a)^n = (z-a)^m \sum_{n=0}^{\infty} c_{n+m} (z-a)^n, \tag{68}$$

where $c_n = f^{(n)}(a)/n!$, and $c_m \neq 0$. An alternative characterization of a zero of order m that can be useful is given by the following:

Lemma 7.12 *Suppose f is analytic on a domain $D \subset \mathbb{C}$. Then f has a zero of order m at $a \in D$ if and only if $\exists\ \delta > 0$ and g analytic and nonzero on $B(a;\delta)$ such that*

$$f(z) = (z-a)^m g(z).$$

Proof $f(z)$ is analytic at $a \in D$ and thus it has a Taylor series expansion about $z = a$ that converges uniformly on the largest disk $B(a;r) \subset D$. If f has a zero of order m at $a \in D$ then by (68) we have

$$f(z) = (z-a)^m g(z), \quad \text{where} \quad g(z) = \sum_{n=0}^{\infty} c_{n+m}(z-a)^n, \quad c_m \neq 0.$$

Clearly $g(a) \neq 0$, and by analyticity of f and convergence of its Taylor series, the series defining g also converges uniformly on $B(a;r)$. By theorem 7.7 then, g is analytic on $B(a;r)$ (the above series representation is in fact its Taylor series about $z = a$). It is thus also continuous at $z = a$, and so given $\epsilon > 0$ $\exists \delta_\epsilon > 0$ such that

$$|g(z) - g(a)| < \epsilon \quad \text{when} \quad |z-a| < \delta_\epsilon.$$

Applying the triangle inequality in the form $|g(z) - g(a)| \geq ||g(z)| - |g(a)||$ we then have

$$-\epsilon < |g(z)| - |g(a)| < \epsilon \quad \text{when} \quad |z-a| < \delta_\epsilon,$$

and so choosing $\epsilon = |g(a)| > 0$, and $\delta = \min\{\delta_\epsilon, r\}$,

$$|g(z)| > 0 \quad \text{when} \quad |z-a| < \delta.$$

This proves the result one way. The converse is easier; if $f(z)$ has the assumed representation in terms of g then the Taylor expansion of g must take the form

$$g(z) = \sum_{n=0}^{\infty} c_{n+m}(z-a)^n, \quad c_m \neq 0,$$

uniformly convergent on some $B(a;\delta)$, which gives

$$f(z) = (z-a)^m g(z) = \sum_{n=m}^{\infty} c_n (z-a)^n, \quad c_m \neq 0$$

as the unique Taylor expansion of f on $B(a;\delta)$. It follows then from the definition 7.11, and from the definition of the Taylor coefficients in (58), that f has a zero of order m at a.

7.2.1 Identity theorem for analytic functions

Once we know that an analytic function has a unique Taylor series expansion about one of its zeros, valid within some disk, a straightforward corollary is the following **identity theorem**:

Theorem 7.13 *(Identity theorem) Suppose $f(a) = 0$ and f is analytic on $B(a;r)$ $(r > 0)$. Then if $Z_{a,r}(f)$, the set of zeros of f in $B(a;r)$, has a* **limit point** *in $B(a;r)$, then f is* **identically zero** *in $B(a;r)$.*

An equivalent statement is, of course, that in such a disk, the zeros of the analytic function f are isolated unless f is identically zero. The theorem has several stronger variants, but we shall prove only this simplest case before stating a stronger form and hand-waving.

Proof of theorem 7.13. By Taylor's theorem, f has the (unique) power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad z \in B(a;r). \tag{69}$$

Then there are two possibilities:
(i) All coefficients c_n are zero, in which case f is identically zero on $B(a;r)$.
(ii) There exists a smallest integer $m > 0$ such that $c_m \neq 0$. The series $g(z) := \sum_{n=0}^{\infty} c_{n+m}(z-a)^n$ has radius of convergence $R \geq r$. To see this, recall theorem 7.7, where it was shown that a differentiated power series has the same radius of convergence as the original power series. We use this result for the power series (69), differentiated m times:

$$f(z) = \sum_{n=0}^{\infty} c_{n+m} (z-a)^{n+m}$$

$$\Rightarrow \quad f^{(m)}(z) = \sum_{n=0}^{\infty} (n+m)(n+m-1)\dots(n+1)c_{n+m}(z-a)^n,$$

together with the comparison test (the terms of $g(z)$ have absolute value less than or equal to those of $f^{(m)}(z)$, which we know converges).

Since $g(a) \neq 0$ and g is continuous at a (theorem 7.2, or the stronger theorem 7.7), $g(z) \neq 0$ for $z \in B(a;\epsilon)$, some $\epsilon > 0$. In the punctured ball $B'(a;\epsilon)$, $f(z) = (z-a)^m g(z) \neq 0$, so a is not a limit point of $Z(f)$.

In conclusion, if a is a zero of f, then **either** $f \equiv 0$ on $B(a;r)$ (some $r > 0$), **or** a cannot be a limit point of zeros (that is, it is an isolated zero). The theorem follows.

More generally, we have:

Theorem 7.14 *(Identity theorem 2) Let f be analytic on a domain D. If $Z_D(f)$, the set of zeros of f in D, has a limit point in D, then $f \equiv 0$ in D.*

Equivalently: the zeros of an analytic function are **isolated** in a domain, unless the function is identically zero. A handwaving proof of this theorem follows the proof of theorem 7.13 above. Let $a \in D$ be a zero of f, then choose $r > 0$ to be the largest number such that $B(a;r) \subset D$ (r always exists since D is open). Then, as above, a is either an isolated zero, or else f is identically zero on $B(a;r)$ (in fact $f \equiv 0$ on $\overline{B(a;r)}$, by continuity). In the latter case, we may choose another zero a_1 of f from the edge of $\overline{B(a;r)}$ and repeat the argument to show that $f \equiv 0$ on the largest ball centered on a_1 and contained within D. Continuing in this way, we cover D with a sequence of overlapping balls on each of which $f \equiv 0$, so that $f \equiv 0$ on D. The only alternative to this is that each zero of f is isolated on D, so that we can never deduce that $f \equiv 0$ on any ball contained within D.

This theorem has a useful corollary for analytic functions that are equal on some subset of their domain of analyticity.

Corollary 7.15 *Let f and g be functions analytic on some common domain D, and suppose that $f(z) = g(z)$ for $z \in S \subset D$, where S contains at least one of its limit points. Then $f(z) \equiv g(z)$ on D.*

Proof Apply the identity theorem to the function $F(z) = f(z) - g(z)$. $F(z)$ is analytic on D, and its set of zeros S contains a limit point. Thus $F(z) \equiv 0$ by theorem 7.14.

We illustrate the results by example.

Example 7.16 *Using the identity theorem, show that there exists a unique analytic function $f(z)$, defined on a domain D containing the origin, such that*

$$f\left(\frac{1}{n}\right) = (n^2 - 1)^{-1} \tag{70}$$

for all but finitely many $n \in \mathbb{N}$. Give $f(z)$.

Suppose there are two such functions, f_1 and f_2. Then $g(z) = f_1(z) - f_2(z)$ has zeros at points $z_n = 1/n$ for all but finitely many $n \in \mathbb{N}$. It follows that 0 is a limit point of zeros of g in D. By the identity theorem 7.14 then, $g \equiv 0$ in D. The function f is therefore unique, and the only such function is

$$f(z) = \frac{z^2}{1 - z^2}$$

(replace n by $1/z$ in (70)).

Example 7.17 *Let $f(z) = u(x, y) + iv(x, y)$ be analytic on $B(0; 1)$. Prove that the function $g(z)$ defined by*

$$g(z) = f(z) - \overline{f(-\bar{z})}$$

is analytic on $B(0; 1)$. Show further that, if f takes real values on the imaginary axis, then

$$u(x, y) = u(-x, y), \quad v(x, y) = -v(-x, y)$$

for all points (x, y) in $B(0; 1)$.

The analyticity of $g(z)$ follows once we've established that $\overline{f(-\bar{z})}$ is analytic on $B(0; 1)$, and this can be done by appealing to the Cauchy-Riemann equations, noting that $\overline{f(-\bar{z})} = \overline{f(-x + iy)} = p(x, y) + iq(x, y)$, where $p(x, y) = u(-x, y)$, $q(x, y) = -v(-x, y)$. The point $(-x, y)$ lies in $B(0; 1)$ if and only if (x, y) does. Checking the Cauchy-Riemann equations for p and q we find that

$$\begin{aligned} p_x(x, y) &= -u_x(-x, y) = -v_y(-x, y) = q_y(x, y), \\ p_y(x, y) &= u_y(-x, y) = -v_x(-x, y) = -q_x(x, y), \end{aligned}$$

where we used the Cauchy-Riemann equations for u and v. Thus, the Cauchy-Riemann equations hold for the real and imaginary parts of $\overline{f(-\bar{z})}$, and so this function is analytic on $B(0;1)$.

For the second part, observe that $z = iy = -\bar{z}$ on the imaginary axis, and thus on this line

$$g(z) = f(iy) - \overline{f(iy)} = 0, \quad -1 < y < 1,$$

since $f(iy) \in \mathbb{R}$. The zeros of the analytic function g are not isolated; therefore by the identity theorem we have $g(z) = 0$ on its domain of analyticity, as required.

The indentity theorem 7.14 (or the uniqueness corollary 7.15) allows certain functional identities to be extended from subsets of $\mathbb{C}$ to larger subsets, or the whole of $\mathbb{C}$.

Example 7.18 *The identity* $\cos^2 z + \sin^2 z = 1$ *holds for all* $z \in \mathbb{C}$.

We know the identity to hold for $z \in \mathbb{R}$ by results from real analysis. Writing $f(z) = \sin^2 z + \cos^2 z - 1$ we have $f(z) \equiv 0$ for $z \in \mathbb{R} \subset \mathbb{C}$. Since $\mathbb{R}$ contains limit points (every point of $\mathbb{R}$ is a limit point of itself), the identity theorem tells us that $f(z) \equiv 0$ on $\mathbb{C}$.

Example 7.19 *(The binomial theorem for negative integers n) Show that, for any negative integer n,*

$$(1+z)^n = \sum_{k=0}^{\infty} \frac{n!}{(n-k)!k!} z^k, \quad |z| \leq 1.$$

We write

$$f(z) = (1+z)^n, \quad g(z) = \sum_{k=0}^{\infty} \frac{n!}{(n-k)!k!} z^k.$$

Clearly f is analytic except at $z = -1$, while the series defining g has radius of convergence 1. By theorem 7.7 g then defines an analytic function on $B(0;1)$. We know from real analysis that $f(z) = g(z)$ when $z \in \mathbb{R}$ with $z| < 1$. The uniqueness corollary 7.15 then tells us that $f \equiv g$ for all $z \in B(0;1)$.

The result embedded in theorems 7.13, 7.14 and 7.15 represent an elementary form of a more general principle known as **analytic continuation**. The basic idea of analytic continuation is to take limited information about an analytic function f (e.g. its values on some small subset, such as a line or curve in $\mathbb{C}$), and use this information to deduce the values of the function in some larger set (even perhaps the whole complex plane). This concept, which will be explored in more detail in Math 756, is linked to our earlier observations about the connection between analytic functions and real "harmonic conjugate" functions (see remarks following the Cauchy Integral formula, theorem 6.1), though it is more general. We observed there that the values of a harmonic function u on some boundary curve are sufficient to determine uniquely its values inside the boundary curve (or even on one side of a semi-infinite boundary curve, provided we impose suitable bounds on the growth at infinity). Once we have the harmonic function then we can determine its harmonic conjugate v, and hence construct the associated analytic function $f = u + iv$. Since we need only the real (or imaginary) part of the analytic function on a curve to construct the whole analytic function away from the curve, it should be no surprise that knowledge of both real **and** imaginary parts are certainly sufficient to "continue" the analytic function away from the curve (or other subdomain in question). In fact if anything, one has to worry about the compatibility of the data on the subset – as we saw earlier, the real and imaginary parts of f cannot be prescribed arbitrarily on a boundary curve or subset of $\mathbb{C}$.

7.3 The Open Mapping theorem, and Maximum principles revisited

The identity theorem proved above can be used to prove an important result known as the Open Mapping Theorem: loosely-speaking, open sets are mapped to open sets by analytic functions. We state and prove the theorem for a function analytic on a domain D:

Theorem 7.20 *(Open Mapping Theorem) If $f(z)$ is analytic and non-constant on a domain $D \subset \mathbb{C}$ (open), then $f(D)$ is open.*

Proof Choose an arbitrary point $a \in D$, then $f(a) \in f(D)$. To show that $f(D)$ is open we need to find a ball centered on $f(a)$ that is contained within $f(D)$, that is, we must find $\epsilon_2 > 0$ such that $B(f(a); \epsilon_2) \subset f(D)$.
By considering $f(z) - f(a)$, we may assume that $f(a) = 0$ without loss of generality. Since f is non-constant (and therefore not identically zero on

D), the identity theorem 7.13 tells us that a is not a limit point of zeros. Therefore, $\exists \delta > 0$ such that $B(a;2\delta) \subset D$ **and** $f(z) \neq 0$ on $B'(a;2\delta)$.
We next note that $f(z)$ is analytic (thus continuous) on the compact set represented by $C(a;\delta)$ ($C(a;\delta) = \{z : |z-a| = \delta\} \subset B(a;2\delta)$); and so it is bounded and attains its bounds on $C(a;\delta)$. We let $2m > 0$ be the **minimum** of $|f(z)|$ on $C(a;\delta)$.
We shall show that $B(f(a) = 0; m) \subset f(D)$, proving that $f(D)$ is open (since $a \in D$ was arbitrary). Let $w \in B'(0;m)$, so $0 < |w| < m$; and for a contradiction, assume that $w \neq f(z)$ for any $z \in D$. Then in particular, $w \neq f(z)$ for any $z \in B(a;2\delta)$, and so the function

$$h(z) = \frac{m}{f(z) - w}$$

is analytic on $B(a;2\delta)$. On $C(a;\delta)$ we have $|f(z) - w| \geq |f(z)| - |w| > 2m - m = m$, and so $|h(z)| \leq 1$. By the maximum principle theorem 6.5 then, $|h(a)| \leq 1$ also. But

$$h(a) = \frac{m}{-w} \quad \Rightarrow \quad |h(a)| > 1,$$

giving the required contradiction. Hence $B(0;m) \subset f(B(a,2\delta)) \subset f(D)$.

Note that this result is not true for infinitely differentiable **real-valued** functions $f(x)$. A simple counterexample is given by the function $f(x) = x^2$, which maps the open interval $(-1,1)$ to $[0,1)$, which is not open.

Example 7.21 *The function $f(z) = z\bar{z}$ takes the unit ball, $B(0;1)$, onto the real interval $0 \leq \Re(f(z)) < 1$, $\Im(f(z)) = 0$. This is not an open subset of $\mathbb{C}$, since we cannot fit an open ball that lies within the set around any point of the set. Therefore f cannot be analytic on $B(0;1)$ (as we already knew).*

It is also true that analytic functions map compact sets to compact sets in $\mathbb{C}$ – but note that we cannot prove this simply by taking complements in the open mapping theorem. While it is true that D^c is closed if and only if D is open, we have no guarantee that $f(D)^c = f(D^c)$; and indeed this is not true in general. For example, the simple function $f(z) = z(z-2)$ takes both points $z = 0$ and $z = 2$ to the origin in the image plane. Therefore for this function, taking $D = B(0;1)$, both $f(D)$ and $f(D^c)$ have a point in common (the origin), and thus they are not complementary sets. We will not prove the result that compact sets are mapped to compact sets by analytic functions.

The open mapping theorem allows us to give a very easy proof of the maximum modulus theorem 6.5 – which was used in proving the open mapping theorem! The maximum modulus theorem (and its variants) has implications for real harmonic functions, as we shall see. We now restate the theorem, in a slightly different form, and give the simple proof.

Theorem 7.22 *(Maximum modulus theorem for analytic functions, 2) (i) If f is a non-constant analytic function on the domain $D \subset \mathbb{C}$ then $|f(z)|$ has no maximum in D.*
(ii) If g is analytic on the bounded domain $D \subset \mathbb{C}$ and continuous on $\overline{D} = D \cup \partial D$ then $\exists z_{max} \in \partial D$ such that $|g(z)| \leq |g(z_{max})| \forall z \in D$.

Proof The open mapping theorem 7.20 guarantees that the image of a domain D under a non-constant analytic function f, $f(D)$, is open. (In fact, $f(D)$ is a domain if D is a domain.)
(i) Let $a \in D$. Then $f(a) \in f(D)$, and $f(D)$, being open, contains a neighborhood of $f(a)$, and therefore a point of larger modulus.
(ii) Since the real function $|g|$ is continuous on the closed bounded set $\overline{D}$, $|g|$ has a maximum at some point $z_{max} \in \overline{D}$,

$$|g(z_{max})| = M = \sup\{|g(z)| : z \in \overline{D}\}.$$

If $z_{max} \in D$ then g is constant by (i) (so it attains its maximum modulus trivially on ∂D). Otherwise $z_{max} \in \partial D$.

Corollary 7.23 *(Minimum modulus principle) If $f(z)$ does not vanish on the domain $D \subset \mathbb{C}$ then $|f|$ attains its minimum value on the boundary ∂D.*

Proof If f does not vanish on D then $g(z) = 1/f(z)$ is analytic on D and so, by theorem 7.22, attains its maximum modulus on the boundary ∂D (or is constant, in which case the result follows trivially).

Corollary 7.24 *(Maximum principle for the Laplace equation) A function $u(x, y)$ harmonic on a domain D attains both its maximum and minimum values on the boundary ∂D.*

Proof Since u is harmonic on D it is the real part of a function $f(z)$ analytic on D. The function $g(z) = \exp(f(z))$ is also analytic and nonvanishing on D, and so attains its maximum modulus and its minimum modulus on the boundary ∂D. Since

$$|g(z)| = |\exp(f(z))| = |\exp(u+iv)| = \exp(u),$$

it follows that $u(x,y)$ must attain its maximum and minimum values on the boundary ∂D. (Note that this last equality above makes it clear that $g(z)$ is nonvanishing on D.)

Note: the desired result can also be deduced from the maximum modulus principle, theorem 7.22, alone, by consideration of the analytic function $\exp(-f(z))$ to show that $u(x,y)$ attains its minimum value on the boundary.

Homework: Ablowitz & Fokas, Problems for Section 2.1, questions 4,5,7.

8 Laurent expansions

Taylor's theorem, giving power-series expansions about a given point z_0, only applies to functions that are analytic in a disk about z_0. If f fails to be analytic at z_0, but is analytic in some **annular region** about z_0, then a generalization of Taylor's theorem may be found, in which **negative** as well as positive powers of $(z-z_0)$ appear in the series expansion. Such a series expansion for a function f is called a **Laurent series**. It is certainly clear that such an expansion can be written down sometimes: consider for example the function $f(z) = e^{z^2}/z^2$. We know that e^{z^2} has a power series expansion about the origin $z=0$ that converges uniformly $\forall z \in \mathbb{C}$, given by $\sum_{n=0}^{\infty} z^{2n}/n!$. Therefore

$$f(z) = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^{2n}}{n!} = \sum_{n=-1}^{\infty}\frac{z^{2n}}{(n+1)!}.$$

The expansion contains negative powers of z and is clearly divergent at $z=0$, as we would expect. It is, however, convergent on any annulus about the origin, if we cut out a small disk around $z=0$. Such an expansion represents a generalization of the Taylor expansion for analytic functions. The following theorem guarantees that such a generalized power series expansion can always be found for functions that are analytic on some annular domain.

8.1 Laurent's theorem

Theorem 8.1 *(Laurent series) A function $f(z)$ analytic in an annulus $R_1 \leq |z - z_0| \leq R_2$ may be written as*

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n \tag{71}$$

in $R_1 \leq |z - z_0| \leq R_2$, where

$$C_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)dw}{(w - z_0)^{n+1}} \tag{72}$$

and γ is any simple closed contour lying within the region of analyticity and enclosing the inner boundary $|z - z_0| = R_1$. This series representation for f is unique, and converges uniformly to $f(z)$ for $R_1 < |z - z_0| < R_2$.

Definition 8.2 *(Residue) The coefficient of the term $(z - z_0)^{-1}$ in the Laurent expansion, C_{-1}, is known as the* **residue** *of the function f at z_0.*

Definition 8.3 *(Principal part) The singular part of the Laurent expansion, $\sum_{n=-\infty}^{-1} C_n (z - z_0)^n$, is known as the* **principal part** *of the Laurent expansion.*

Remark In the case that f is analytic on the whole disc $|z| \leq R_2$ it is easily checked, using Cauchy's formula for derivatives (55) for $n \geq 0$, and Cauchy's theorem 5.10 for $n < 0$, that the Laurent expansion (71) for f reduces to the Taylor expansion (58).

Proof (of theorem 8.1) We take circular contours $\gamma_1 = C(z_0; R_1)$ and $\gamma_2 = C(z_0; R_2)$, and introduce a cut between them. Denoting by γ_{12} the cut followed from γ_1 to γ_2, and by γ_{21} the cut from γ_2 to γ_1, f is analytic on the region D enclosed by the contour $\gamma = \gamma_2 \cup \gamma_{21} \cup \gamma_1 \cup \gamma_{12}$. Applying Cauchy's integral theorem 6.1 to $f(z)$ on this region, the line integrals along the two cuts cancel (being taken in opposite directions) and thus for any point z within D we have

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(w)\, dw}{w - z} - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(w)}{w - z}. \tag{73}$$

In the first integral we have $|z - z_0| < |w - z_0|$ for any $w \in \gamma_2$, and thus

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{(w-z_0)\left(1-\frac{z-z_0}{w-z_0}\right)} \\ &= \frac{1}{w-z_0}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^j, \end{aligned} \tag{74}$$

where the sum is uniformly convergent. In the second integral (around γ_1) in (73), on the other hand, we have $|w - z_0| < |z - z_0|$ for any $w \in \gamma_1$, and thus

$$\begin{aligned} -\frac{1}{w-z} &= \frac{1}{(z-z_0)-(w-z_0)} = \frac{1}{(z-z_0)\left(1-\frac{w-z_0}{z-z_0}\right)} \\ &= \frac{1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^j; \end{aligned} \tag{75}$$

again the sum converges uniformly. Using (74) and (75) in (73) gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{\gamma_2}\sum_{j=0}^{\infty}\frac{(z-z_0)^j}{(w-z_0)^{j+1}}f(w)\,dw + \\ &\quad \frac{1}{2\pi i}\oint_{\gamma_1}\sum_{j=0}^{\infty}\frac{(w-z_0)^j}{(z-z_0)^{j+1}}f(w)\,dw. \end{aligned} \tag{76}$$

The sums and integrals can be interchanged, by theorem 7.3. We also relabel in the second sum, writing $j = -(m+1)$, for m running from $-\infty$ to -1. Then (76) becomes

$$\begin{aligned} f(z) &= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(w)\,dw}{(w-z_0)^{j+1}}\right)(z-z_0)^j + \\ &\quad \sum_{m=-\infty}^{-1}\left(\frac{1}{2\pi i}\oint_{\gamma_1}\frac{f(w)\,dw}{(w-z_0)^{m+1}}\right)(z-z_0)^m. \end{aligned}$$

Finally, noting that the integrand in all cases is analytic within D, we can use theorem 5.11 to replace each of γ_1 and γ_2 by an arbitrary contour lying between γ_1 and γ_2, giving

$$f(z) = \sum_{j=-\infty}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma}\frac{f(w)\,dw}{(w-z_0)^{j+1}}\right)(z-z_0)^j \equiv \sum_{j=-\infty}^{\infty} C_j(z-z_0)^j,$$

as claimed.

Remark From the form of the Laurent expansion and our knowledge of the properties of power series, it is clear that the function f is analytic at the point z_0 if and only if all the Laurent coefficients C_j for $j < 0$ are zero. If some $C_j \neq 0$ for $j < 0$ then f is unbounded at z_0, and cannot be analytic there. For such singular functions, the Laurent expansion can be used to **classify** the type of singularity, as we discuss in §8.2 below.

We first prove that the Laurent expansion derived above is unique.

Theorem 8.4 *(Uniqueness of Laurent expansion) Suppose f is analytic on the annulus $R_1 \leq |z - z_0| \leq R_2$, and that it has a uniformly convergent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} B_n (z - z_0)^n, \quad R_1 \leq |z - z_0| \leq R_2. \tag{77}$$

Then $B_n = C_n$ for all n, where C_n is defined by (72).

Proof Assume $z_0 = 0$ for simplicity (the proof easily generalizes). Let $R_1 < r < R_2$, then

$$\begin{aligned} 2\pi i C_n &= \oint_{C(0;r)} f(w) w^{-(n+1)} dw = \oint_{C(0;r)} \sum_{k=-\infty}^{\infty} B_k w^{k-n-1} dw \\ &= \oint_{C(0;r)} \sum_{k=0}^{\infty} B_k w^{k-n-1} dw + \oint_{C(0;r)} \sum_{m=1}^{\infty} B_{-m} w^{-m-n-1} dw. \end{aligned}$$

By the results on uniform convergence of power series (theorem 7.2 and its corollary 7.3) summation and integration can be interchanged in each of these two power series to give

$$2\pi i C_n = \sum_{k=-\infty}^{\infty} B_k \int_{C(0;r)} w^{k-n-1} dw = 2\pi i B_n$$

(it is easily shown by parametric integration that $\int_{C(0;r)} w^{k-n-1} dw = 2\pi i \delta_{kn}$).

This theorem is useful because in general it is very difficult to evaluate the Laurent coefficients using the formula (72) derived in the theorem 8.1. However, the uniqueness result assures us that if by any other means we can find a valid Laurent expansion for the given function f, then by uniqueness, this expansion must be the unique Laurent expansion for f.

We now consider a few examples to illustrate.

Example 8.5 *Discuss Laurent expansions of the function* $f(z) = 1/[z(1-z)]$.

This function has singularities at $z = 0$ and at $z = 1$, so f will not be analytic in any disk containing either of these points. Considering first the singular point at $z = 0$, the function is analytic in two distinct annuli: $A_1 = \{z : 0 < |z| < 1\}$, and $A_2 = \{z : |z| > 1\}$. We can write f as

$$f(z) = \frac{1}{z} + \frac{1}{1-z},$$

and the second term here is actually analytic on A_1, with power series expansion given by the usual binomial representation; therefore

$$f(z) = z^{-1} + \sum_{n=0}^{\infty} z^n.$$

By uniqueness, this must be the Laurent series for f on A_1:

$$f(z) = \sum_{n=-1}^{\infty} z^n, \quad z \in A_1.$$

On A_2 we have

$$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -z^{-1}(1-z^{-1})^{-1},$$

and the final bracketed term here can again be binomially expanded since $|z|^{-1} < 1$ on A_2. Then

$$f(z) = \frac{1}{z} + \frac{1}{1-z} = z^{-1} - z^{-1}\sum_{n=0}^{\infty} z^{-n} = -\sum_{k=-\infty}^{-2} z^k, \quad z \in A_2.$$

If we had not noted the partial fraction decomposition we could still obtain the Laurent expansion quite easily, noting that on A_1

$$f(z) = \frac{1}{z(1-z)} = z^{-1}(1-z)^{-1} = z^{-1}\sum_{n=0}^{\infty} z^n = \sum_{n=-1}^{\infty} z^n,$$

uniformly convergent for $z \in A_1$; and on A_2,

$$f(z) = \frac{1}{z(1-z)} = z^{-1}(1-z)^{-1} = -z^{-2}(1-z^{-1})^{-1} = -z^{-2}\sum_{n=0}^{\infty} z^{-n} = \sum_{n=-\infty}^{-2} z^n,$$

uniformly convergent for $z \in A_2$.

We can also find Laurent expansions about the singular point $z = 1$, noting that f is analytic in annuli A_3 $(0 < |z-1| < 1)$, and A_4 $(|z-1| > 1)$ about $z = 1$. We can either start from the partial fraction decomposition, or take the direct approach as above. Then on the annulus A_3,

$$\begin{aligned} f(z) &= \frac{1}{z(1-z)} = -(z-1)^{-1}(1+(z-1))^{-1} \\ &= -(z-1)^{-1}\sum_{n=0}^{\infty}(-1)^n(z-1)^n \\ &= \sum_{n=-1}^{\infty}(-1)^n(z-1)^n, \end{aligned}$$

uniformly convergent for $z \in A_3$. For $z \in A_4$, the direct approach gives

$$\begin{aligned} f(z) &= \frac{1}{z(1-z)} = -(z-1)^{-1}(1+(z-1))^{-1} \\ &= -(z-1)^{-2}(1+(z-1)^{-1})^{-1} \\ &= -(z-1)^{-2}\sum_{n=0}^{\infty}(-1)^n(z-1)^{-n} \\ &= \sum_{n=-\infty}^{-2}(-1)^{n+1}(z-1)^n \end{aligned}$$

uniformly convergent on $|z-1| > 1$ (region A_4).

Example 8.6 *Find a Laurent series expansion for the function* $\tan z$*, about* $z = \pi/2$.

Writing $u = z - \pi/2$ we have

$$\begin{aligned}
f(z) &= \frac{\sin z}{\cos z} = \frac{\sin(\pi/2+u)}{\cos(\pi/2+u)} = -\frac{\cos u}{\sin u} = -\frac{\sum_{n=0}^{\infty}\frac{(-1)^n u^{2n}}{(2n)!}}{\sum_{n=0}^{\infty}\frac{(-1)^n u^{2n+1}}{(2n+1)!}} \\
&= -\frac{1}{u}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n u^{2n}}{(2n)!}}{\sum_{n=0}^{\infty}\frac{(-1)^n u^{2n}}{(2n+1)!}} \\
&= -\frac{1}{u}\left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots\right)\left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} + \cdots\right)^{-1} \\
&= -\frac{1}{u}\left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots\right)\left(1 + \frac{u^2}{3!} + u^4\left(-\frac{1}{5!} + \frac{1}{(3!)^2}\right) + \cdots\right) \\
&= -\frac{1}{u}\left(1 - \frac{u^2}{3} + u^4\left(\frac{1}{(3!)^2} - \frac{1}{5!} + \frac{1}{4!} - \frac{1}{2!3!}\right) + \cdots\right).
\end{aligned}$$

Setting $u = z - \pi/2$ in here we obtain the desired Laurent expansion.

Example 8.7 *Find a Laurent series expansion of the function $f(z) = e^{-1/z^2}$.*

Considered as a real function of a real variable $z = x$, although (strictly speaking) it is singular at $x = 0$, this function can be plotted for all real x and is actually very smooth, approaching zero exponentially as $x \to 0$. However in the complex plane it is very badly-behaved as $z \to 0$, because if we consider $z = iy$, $y \to 0$, we have $-1/z^2 = 1/y^2 \to +\infty$, so that e^{-1/z^2} blows up exponentially as $z \to 0$ along the imaginary axis. This bad behavior is reflected in its Laurent series expansion which, for $z \neq 0$, can be obtained from the usual power series representation of the exponential function:

$$e^{-1/z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n}}{n!} = \sum_{n=-\infty}^{0} \frac{(-1)^n z^{2n}}{|n|!},$$

uniformly convergent for any $z \neq 0$. Hence this expansion is valid for $|z| > R$ for any $R > 0$.

Example 8.8 *Find a Laurent series expansion of the function*

$$A(z) = \int_z^{\infty} \frac{e^{-1/w}}{w^2} dw \tag{78}$$

about $z = 0$, valid on $|z| > R$, $R > 0$.

The function $e^{-1/w}$ in the integrand has a Laurent expansion convergent on any $|z| > R$ as above:

$$e^{-1/w} = \sum_{n=0}^{\infty} \frac{(-1)^n w^{-n}}{n!},$$

and thus

$$\frac{e^{-1/w}}{w^2} = \sum_{n=0}^{\infty} \frac{(-1)^n w^{-(n+2)}}{n!}$$

also uniformly convergent on $|z| > R$ for any $R > 0$. By theorem 7.3 then, we can interchange the order of summation and integration in the definition of $A(z)$ and integrate term-by-term, to find

$$\begin{aligned} A(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_z^{\infty} w^{-(n+2)}\, dw \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} [w^{-(n+1)}]_z^{\infty} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} z^{-(n+1)} \\ &= \sum_{-\infty}^{-1} \frac{(-1)^{n+1}}{|n|!} z^n. \end{aligned}$$

Looking at the third equality above we see that

$$A(z) = -\sum_{1}^{\infty} \frac{(-1)^n}{n!} z^{-n} = 1 - \sum_{0}^{\infty} \frac{(-1)^n}{n!} z^{-n} = 1 - e^{-1/z}, \qquad (79)$$

suggesting that we could have evaluated the integral explicitly; and indeed we see that differentiation of each of (78) and (79) leads to the same result, $A'(z) = -e^{-1/z}/z^2$.

This procedure does not always work – for example, consider the function $E(z)$ defined by

$$E(z) = \int_z^{\infty} \frac{e^{-w}}{w} dw.$$

The integral is convergent, but the above procedure leads to divergent integrals.

Homework: Ablowitz & Fokas, problems for §3.3, questions 2,3,5.

8.2 Singular points

We begin by discussing the simplest kind of functional singularities: where the function $f(z)$ is analytic on some punctured disk about z_0 ($0 < |z - z_0| < R$ for some $R > 0$), but is not analytic at z_0 itself. Such a singular point z_0 is known as an **isolated singular point** of f. (Branch points are not isolated singular points.) There are several types of such isolated singular points.
(i) If z_0 is an isolated singular point of f at which $|f|$ is bounded (*i.e.* there is some $M > 0$ such that $|f(z)| \leq M$ for all $|z| \leq R$) then z_0 is a **removable singularity** of f. Clearly, all coefficients C_n with $n < 0$ must be zero in the Laurent expansion (71), thus f in fact has a regular power series expansion $f(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n$ valid for $0 < |z - z_0| < R$. Since this power series converges at $z = z_0$ it follows that if we simply redefine $f(z_0) = C_0$ then f is analytic on the whole disc $|z - z_0| < R$, with Taylor series $f(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n$.

Example 8.9 *The function $f(z) = \sin z/z$ has a removable singularity at $z = 0$. Strictly speaking this function is undefined at zero, but it has Laurent series*

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}, \quad |z| > 0,$$

so $z = 0$ is a removable singularity of f, which we remove by defining $f(0) = 1$.

Example 8.10 *The function*

$$f(z) = \frac{e^{z^2} - 1 - z^2}{z^4}$$

has a removable singularity at $z = 0$. It is undefined at $z = 0$, but has Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n+2)!}, \quad |z| > 0,$$

so redefining $f(0) = 1$ removes the singularity.

(ii) If $f(z)$ can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^N}$$

where N is a positive integer and $g(z)$ is analytic on $|z - z_0| < R$ with $g(z_0) \neq 0$, then the point z_0 is a **pole of order** N (a **simple pole** if $N = 1$). Clearly $f(z)$ is unbounded as $z \to z_0$.

Example 8.11 *The function*

$$f(z) = \frac{e^{2z} - 1}{z^2}$$

has a simple pole at $z = 0$.

Its Laurent expansion is given by

$$f(z) = \sum_{n=-1}^{\infty} \frac{2^{(n+2)} z^n}{(n+2)!},$$

so the residue at $z = 0$ is $C_{-1} = 2$.

Example 8.12 *The function $f(z) = 1/\sinh z$ has a simple pole at $z = 0$.*

The Laurent expansion is found from inverting the Taylor expansion of $\sinh z$:

$$\begin{aligned} f(z) &= \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)^{-1} = \frac{1}{z}\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)^{-1} \\ &= \frac{1}{z}\left(1 - \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^4}{(3!)^2} + \cdots\right), \end{aligned}$$

so the residue at $z = 0$ is 1.

Functions having poles as their only singularities are known as **meromorphic**.

(iii) An isolated singular point that is neither removable nor a pole is called an **essential singular point.** The Laurent expansion of the function about such a singular point is non-terminating for $n < 0$, that is, there is no positive integer N such that $C_{-n} = 0$ for all $n > N$.

Example 8.13 *The function*

$$f(z) = e^{1/z^2} \quad \textit{with Laurent expansion} \quad f(z) = \sum_{n=-\infty}^{0} \frac{z^{2n}}{|n|!}$$

has an essential singularity at $z = 0$. It is analytic on the rest of the complex plane, and the Laurent series converges uniformly everywhere except $z = 0$.

Other types of non-isolated singularities include **branch points** and **cluster points**. From the definition 3.26 we already know that a multi-function with branch points cannot be analytic in any punctured disk surrounding the branch point (although sometimes an analytic branch of the function may be constructed in an annular region surrounding two or more branch points – recall, for example, the composite square-root function $f(z) = [(z - a)(z - b)]^{1/2}$ considered in §3.6.1).

A function with a cluster point singularity is one that is singular at points $a_n \in \mathbb{C}$, where a_n is a convergent complex sequence. The limit of the sequence will be a cluster point singularity of the function.

Homework: Review your notes on branch points, branch cuts and multi-functions.

The **residue** of a function at a singular point is very important in contour integral applications, as we're about to see. Therefore we need ways to evaluate this quantity quickly and easily for a given function. The following theorem gives one easy way to extract the residue at a singularity:

Theorem 8.14 *(Evaluating residues at poles) If $f(z)$ has a pole of order k at $z = z_0$ then*

$$Res(f(z), z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}((z - z_0)^k f(z)) \tag{80}$$

Proof Direct manipulation of the Laurent expansion of f about z_0,

$$f(z) = \sum_{n=-k}^{\infty} C_n (z - z_0)^n$$

yields the result for $\mathrm{Res}(f(z), z_0) = C_{-1}$.

Homework: Ablowitz & Fokas, problems for section 2.1, question 3.
Ablowitz & Fokas, problems for section 3.2, question 2(b),(f), 6(c).
Ablowitz & Fokas, problems for section 3.3, question 4.

9 Cauchy's Residue Theorem

9.1 Preliminaries: Motivation

We have seen that Cauchy's theorem is a very powerful result that applies to functions analytic on a domain in $\mathbb{C}$. However, we also now know that many functions are not analytic, and have singularities of various kinds within the domain of interest. We need to be able to evaluate integrals involving non-analytic functions too (after all, it would be quite boring if the answer were always zero). Cauchy's residue theorem enables us to evaluate many such integrals around closed contours, sometimes even where functions have infinitely many singularities within the contour of integration.

There are many reasons why we might want to perform integration of complex functions around contours in the complex plane. A very common application of such contour integration arises in the inversion of Laplace (and Fourier) transforms, which can be used to solve many ordinary and partial differential equations (and which are considered in detail in Math 756). A more commonplace application of complex contour integration is in the evaluation of certain real integrals that are hard to do by standard methods from real calculus. We begin with an example of a real integral that can be reduced to a contour integral.

Example 9.1 *Evaluate* $I = \int_0^\infty (1+x^4)^{-1} dx$.

Method 1: Try to think of a clever (real) substitution that reduces this integral to one we can evaluate explicitly.
Method 2: Try to rewrite the (real) integral as a complex contour integral. Methods from complex analysis can often be used to evaluate such integrals quickly and easily.

Of course, we shall focus on method 2 here. Note that, considered as a complex function, $f(z) = 1/(1+z^4)$, the integrand has singularities (simple poles, in fact) at the points $z = e^{i(2k+1)\pi/4}$ $(k = 0, 1, 2, 3)$ in $\mathbb{C}$, so we should make sure any contour of integration avoids these points. We note also that by symmetry we have

$$I = \lim_{R\to\infty} \int_0^R f(x)dx = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^R f(x)dx;$$

and, recalling example 4.9, we would expect the integral of $f(z)$ along a large circular arc to go to zero as the arc radius $R \to \infty$. We therefore consider $f(z)$ integrated around a semicircular contour γ composed of the real axis from $-R$ to R (this part will correspond to the integral we want), and a semicircle in the upper half plane (this part will go to zero as $R \to \infty$); see figure 4. Parametrizing the semicircular contour in the usual way, $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, we have

$$\lim_{R\to\infty} \int_\gamma f(z)\, dz = 2\int_0^\infty f(x)\, dx + \lim_{R\to\infty} \int_0^\pi \frac{iRe^{i\theta} d\theta}{1 + R^4 e^{4i\theta}}.$$

For the second term on the right-hand side we have

$$\left| \int_0^\pi \frac{iRe^{i\theta} d\theta}{1 + R^4 e^{4i\theta}} \right| \leq \int_0^\pi \frac{R}{R^4 - 1} d\theta = \frac{\pi R}{R^4 - 1} \to 0 \quad \text{as } R \to \infty.$$

Hence,

$$\int_0^\infty \frac{dx}{1 + x^4} = \frac{1}{2} \lim_{R\to\infty} \int_\gamma \frac{dz}{1 + z^4},$$

so if we can evaluate this contour integral then we have the value of the real integral. The contour integral is not zero; Cauchy's theorem does not apply to this case because the contour contains singularities at $z = e^{i\pi/4}$, $e^{3\pi i/4}$. Cauchy's Residue theorem will tell us how to evaluate such integrals.

Similar tricks can be used to convert real integrals involving trigonometric functions to complex integrals around closed contours, in many instances. Again, the procedure is best illustrated by example, but often the trick is to consider an appropriate function integrated around the unit circle, using the fact that $z = e^{i\theta}$ on such a circle.

Example 9.2 *Use a complex substitution to evaluate* $I = \int_0^{2\pi} (1+8\cos^2\theta)^{-1}$.

Note that on the contour, $z = e^{i\theta}$, $z^{-1} = e^{-i\theta}$, and thus $2\cos\theta = z + z^{-1}$. Also, $dz = ie^{i\theta} d\theta$, and so $d\theta = -i dz/z$. Then

$$I = \oint_{C(0;1)} \frac{-i\, dz}{z(1 + 2z^2 + 4 + 2/z^2)} = -i \oint_{C(0;1)} \frac{z\, dz}{2z^4 + 5z^2 + 2}.$$

If the complex integrand here was analytic on the unit disk (within the contour) then Cauchy's theorem would give a zero result immediately. However,

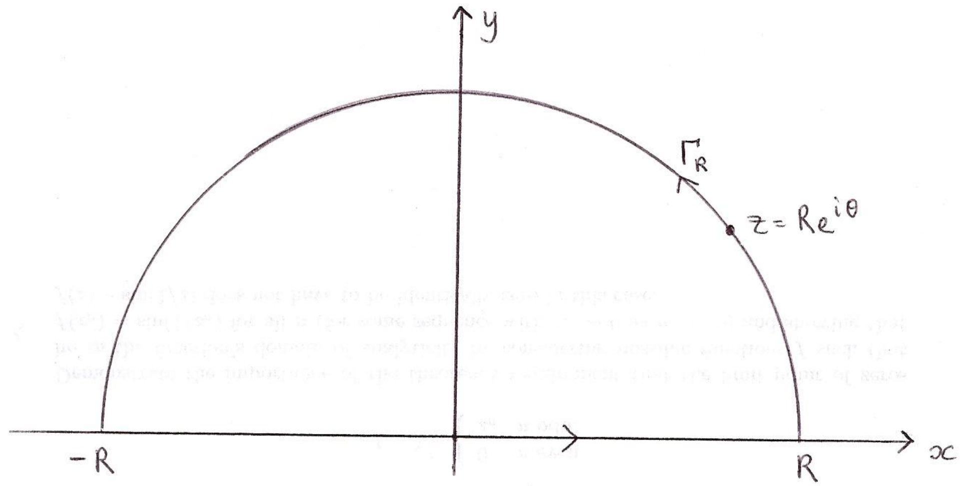

Figure 4: The semicircular contour can be used to evaluate certain real integrals from $-\infty$ to ∞, or from 0 to ∞, as the circle radius $R \to \infty$.

the denominator factorizes as $2z^4 + 5z^2 + 2 = (z^2 + 2)(2z^2 + 1)$, giving singularities within C at $z = \pm i/\sqrt{2}$. Again, if we know how to evaluate contour integrals of functions with pole singularities, then we can evaluate the real trigonometric integral. [From our discussion of Cauchy's integral formula (54) and Cauchy's formula for derivatives (55) we know that we might be able to evaluate the integral by decomposing the rational function as partial fractions and applying those results, but this is unwieldy and we need a more general method.]

9.2 The residue theorem

Such examples demonstrate (in part) the utility of evaluating complex contour integrals with singularities. Cauchy's residue theorem enables us to do exactly that, for integrands that contain isolated singularities within the integration contour.

We first state and prove a simple version of the theorem, framed as the following lemma:

Lemma 9.3 *(Function with a single pole within γ) Let f be analytic inside* *and on a simple closed (positively-oriented) contour γ, except at $z_0 \in \gamma$ where it has an isolated singularity, with Laurent expansion*

$$f(z) = \sum_{-\infty}^{\infty} C_n (z - z_0)^n.$$

Then

$$\oint_\gamma f(z)\, dz = 2\pi i C_{-1}.$$

Proof Choose $r > 0$ such that $\overline{B(z_0; r)} \subset I(\gamma)$. Then

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int_{C(z_0;r)} f(z)\, dz \quad \text{by the deformation theorem 5.11} \\
&= \int_{C(z_0;r)} \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n\, dz \\
&= \sum_{n=-\infty}^{\infty} C_n \int_{C(z_0;r)} (z - z_0)^n\, dz \quad \text{by uniform convergence theorem 7.3} \\
&= \sum_{n=-\infty}^{\infty} C_n \int_0^{2\pi} r^n e^{in\theta} i r e^{i\theta} d\theta \\
&= \sum_{n=-\infty}^{\infty} C_n r^{n+1} (2\pi i \delta_{n,-1}) \\
&= 2\pi i C_{-1}.
\end{aligned}$$

Cauchy's residue theorem is just a simple extension of this result.

Theorem 9.4 *(Cauchy's Residue Theorem) Let f be analytic inside and on a positively oriented contour γ, except for a finite number of isolated singularities at points $z_1, \ldots z_m \in I(\gamma)$. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{m} Res(f(z); z_k).$$

Proof Let f_k be the principal part of the Laurent expansion of f about each z_k. Then

$$g(z) = f(z) - \sum_{k=1}^{m} f_k(z)$$

has only removable singularities at each z_k and so we remove them to define an analytic function on $\overline{I(\gamma)}$. By Cauchy's theorem 5.1, $\int_\gamma g(z)\, dz = 0$, and thus

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{m} \int_\gamma f_k(z)\, dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}(f(z); z_k),$$

by lemma 9.3 applied to each f_k.

9.3 Applications of the theorem to simple integrals

Returning to our earlier examples, we can now evaluate the required integrals. In example 9.1 we need to evaluate

$$\int_\gamma \frac{dz}{1+z^4},$$

where γ is a large semicircular contour in the upper half plane. We know that the integrand is singular at $z = e^{i\pi/4}$ and $z = e^{3\pi i/4}$ inside the contour; once we have evaluated the residue at these points we are done. Writing

$$f(z) = \frac{1}{(z - e^{i\pi/4})(z - e^{3\pi i/4})(z - e^{-3\pi i/4})(z - e^{-i\pi/4})}$$

and using theorem 8.14 we easily find

$$\begin{aligned}
\operatorname{Res}(f(z); e^{i\pi/4}) &= \frac{1}{(e^{i\pi/4} - e^{3i\pi/4})(e^{i\pi/4} - e^{-3i\pi/4})(e^{i\pi/4} - e^{-i\pi/4})} \\
&= \frac{2\sqrt{2}}{((1+i) - (-1+i))((1+i) - (-1-i))((1+i) - (1-i))} \\
&= \frac{2\sqrt{2}}{2.2(1+i)2i} \times \frac{i(1-i)}{i(1-i)} \\
&= \frac{i\sqrt{2}(1-i)}{-8},
\end{aligned}$$

and

$$
\begin{aligned}
\operatorname{Res}(f(z); e^{3i\pi/4}) &= \frac{1}{(e^{3i\pi/4} - e^{i\pi/4})(e^{3i\pi/4} - e^{-3i\pi/4})(e^{3i\pi/4} - e^{-i\pi/4})} \\
&= \frac{2\sqrt{2}}{((-1+i) - (1+i))((-1+i) - (-1-i))((-1+i) - (1-i))} \\
&= \frac{2\sqrt{2}}{-2.2i.(-2)(1-i)} \times \frac{i(1+i)}{i(1+i)} \\
&= \frac{i\sqrt{2}(1+i)}{-8}.
\end{aligned}
$$

Thus by Cauchy's residue theorem 9.4 we have

$$
\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_\gamma \frac{dz}{1+z^4} = \pi i \left(\frac{i\sqrt{2}(1-i)}{-8} + \frac{i\sqrt{2}(1+i)}{-8} \right) = \frac{\pi\sqrt{2}}{4}.
$$

For our second example 9.2 we need

$$
\int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta} = 2\pi \left(\operatorname{Res}(f(z); i/\sqrt{2}) + \operatorname{Res}(f(z); -i/\sqrt{2}) \right),
$$

where $f(z) = z/(2z^4 + 5z^2 + 2) = z/((z^2+2)(2z^2+1))$. Writing

$$
f(z) = \frac{z}{(z^2+2)(\sqrt{2}z + i)(\sqrt{2}z - i)}
$$

and using theorem 8.14 we find

$$
\operatorname{Res}(f(z); i/\sqrt{2}) = \frac{i/\sqrt{2}}{(3/2)(2i)} = \frac{1}{3\sqrt{2}},
$$

and

$$
\operatorname{Res}(f(z); -i/\sqrt{2}) = \frac{-i/\sqrt{2}}{(3/2)(-2i)} = \frac{1}{3\sqrt{2}},
$$

so that finally

$$
\int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta} = \frac{2\pi\sqrt{2}}{3}.
$$

When using complex contour integration to evaluate real integrals, the choice of complex integration (function or contour) to make is not always absolutely straightforward, as the following examples show.

Example 9.5 *Use complex contour integration and the residue theorem to evaluate*

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+x+1} dx.$$

The denominator here, considered as a complex function, has zeros at $(-1 \pm i\sqrt{3})/2$, one in the upper half plane and one in the lower. Our first thought might be to use the same large semicircular contour as above (figure 4), simply replacing $f(x)$ in the integrand by $f(z)$, hoping that the integral along the large circular arc Γ_R again tends to zero as $R \to \infty$ so that the contour integral gives the result we need. However, if we take $f(z) = \cos z/(z^2+z+1)$ then on the arc $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ $(0 \leq \theta \leq \pi)$ we have

$$\begin{aligned} 2|f(z)| &= \left| \frac{e^{iR(\cos\theta + i\sin\theta)} + e^{-iR(\cos\theta+i\sin\theta)}}{R^2e^{2i\theta} + Re^{i\theta} + 1} \right| \\ &\geq \frac{e^{R\sin\theta} - e^{-R\sin\theta}}{R^2+R+1} \\ &\sim \frac{e^{R\sin\theta}}{R^2} \quad \text{as } R \to \infty, \end{aligned}$$

and the integral does not go to zero as the semicircle goes to infinity, but diverges exponentially. Taking a semicircle in the lower half-plane with $\pi \leq \theta \leq 2\pi$ does not help either, since then the other exponential dominates in the numerator. So the "obvious" choice does not work. (The same is true of functions $f(z)$ containing the sine function in the numerator, along large circular arcs.) However, if instead we take

$$f(z) = \frac{e^{iz}}{z^2+z+1},$$

whose real part, when $z = x \in \mathbb{R}$, gives the integrand we need, then on Γ_R we have

$$\begin{aligned} |f(z)| &= \left| \frac{e^{iR(\cos\theta+i\sin\theta)}}{R^2e^{2i\theta} + Re^{i\theta} + 1} \right| \\ &\leq \frac{e^{-R\sin\theta}}{R^2-R-1} \\ &\leq \frac{1}{R^2-R-1} \quad \text{on } \Gamma_R, \end{aligned}$$

and thus for this choice of $f(z)$,

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \leq \int_{\Gamma_R} \frac{R}{R^2-R-1} d\theta \to 0 \quad \text{as } R \to \infty.$$

Putting this together, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+x+1} &= \lim_{R\to\infty} \Re\left(\int_\gamma \frac{e^{iz}\,dz}{z^2+z+1}\right) \\ &= \Re\left(2\pi i \operatorname{Res}(f(z);(-1+i\sqrt{3})/2)\right).\end{aligned}$$

Using theorem 8.14, with the denominator of f factored as $(z+(1+i\sqrt{3})/2)(z+(1-i\sqrt{3})/2)$, we have

$$\begin{aligned}\operatorname{Res}(f(z);(-1+i\sqrt{3})/2) &= \left.\frac{e^{iz}}{z+(1+i\sqrt{3})/2}\right|_{z=(-1+i\sqrt{3})/2} \\ &= \frac{e^{-\sqrt{3}/2}(\cos(1/2)-i\sin(1/2))}{i\sqrt{3}},\end{aligned}$$

and so

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1+1} = \Re\left(\frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}/2}(\cos(1/2)-i\sin(1/2))\right) = \frac{2\pi}{\sqrt{3}}\cos(1/2)e^{-\sqrt{3}/2}.$$

9.4 Jordan's inequality and Jordan's lemma

This is a very useful inequality from real analysis, that can often be applied to contour integrals (as we shall see).

Lemma 9.6 *(Jordan's inequality) Let $\theta \in (0,\pi/2]$. Then*

$$\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1. \tag{81}$$

This inequality can be proved by taking a circle $C(O;1)$, of unit radius and center O, and considering the geometry of a point P in the upper right quadrant, where OP makes an angle θ with the horizontal diameter of the circle, as sketched in figure 5. The perpendicular from P meets the horizontal diameter at a point M, and the circle $C(M;\sin\theta)$ with center M and passing through P is also constructed (of radius $\sin\theta$). If A is the intersection of

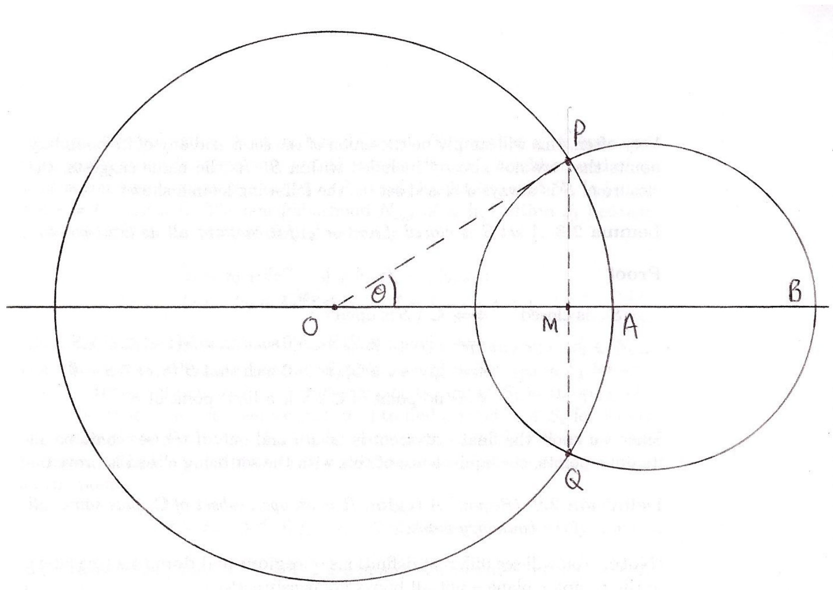

Figure 5: Geometric construction used in the proof of Jordan's inequality.

$C(O;1)$ with the horizontal diameter, and B is the intersection of $C(M,\sin\theta)$ with the horizontal diameter, and Q is the reflection of P in this horizontal, then the inequality is proved by noting that the length of the straight line PQ $(2\sin\theta)$ is less than or equal to the arc length $PAQ = 2\theta$, and the arc length PAQ is less than or equal to the arc length $PBQ = \pi\sin\theta$ (half the circumference of circle $C(M;\sin\theta)$).

This inequality can be used to prove a general result about integrals of analytic functions along semicircular contours which, as we are seeing, arise frequently in applications of the residue theorem.

Lemma 9.7 *(Jordan's Lemma) Suppose $f(z)$ is a complex function that tends to zero, uniformly in z, on the semicircular arc Γ_R in the upper half plane:*

$$f(z) \to 0 \quad \textit{for } z \in \Gamma_R \textit{ as } R \to \infty, \textit{ where } \Gamma_R = \{z : z = Re^{i\theta},\ 0 \le \theta \le \pi\}. \tag{82}$$

Then

$$\lim_{R\to\infty} \int_{\Gamma_R} e^{ikz} f(z)\, dz = 0 \quad (k > 0). \tag{83}$$

Proof The condition on the limiting behavior of f means that on Γ_R ($z = Re^{i\theta}$) we have $|f(z)| \leq M_R$, where M_R is independent of θ but depends on R, and $M_R \to 0$ as $R \to \infty$. Defining

$$I = \int_{\Gamma_R} e^{ikz} f(z)\, dz$$

we then have $z = Re^{i\theta}$, $dz = iRe^{i\theta}$, and $e^{ikz} = e^{-kR\sin\theta + ikR\cos\theta}$, so that

$$|I| \leq \int_0^{\pi} e^{-kR\sin\theta} M_R R\, d\theta = 2M_R R \int_0^{\pi/2} e^{-kR\sin\theta}\, d\theta.$$

We now apply Jordan's inequality, $\sin\theta \geq 2\theta/\pi$, which holds on the range of integration, to deduce that $e^{-kR\sin\theta} \leq e^{-2kR\theta/\pi}$ for $0 \leq \theta \leq \pi/2$. Thus

$$|I| \leq 2M_R R \int_0^{\pi/2} e^{-2kR\theta/\pi}\, d\theta = \frac{M_R \pi}{k}(1 - e^{-kR}) \to 0 \quad \text{as } R \to \infty,$$

because $M_R \to 0$ as $R \to \infty$.

Remark Note that if $k < 0$ in the integral then a similar result holds for the contour Γ_R^-, the semicircle in the **lower** half plane.

Remark If the condition (82) in the statement of Jordan's lemma 9.7 is replaced by the stronger assumption $|zf(z)| \to 0$ as $|z| \to \infty$, then we do not need to invoke the Lemma, as a straightforward estimate then gives the result: since $|e^{ikz}| = e^{-kRsin\theta} \leq 1$ (for $k > 0$, $0 \leq \theta \leq \pi$) we can say

$$\left| \int_{\Gamma_R} e^{ikz} f(z)\, dz \right| \leq |f(Re^{i\theta})| 2\pi R \to 0 \quad \text{as } R \to \infty.$$

Observe, e.g., that we did not need Jordan's lemma in example 9.5 (evaluation of $\cos x/(x^2 + x + 1)$ from $x = -\infty$ to ∞), because the function $1/(z^2 + z + 1)$ decayed sufficiently fast as $|z| \to \infty$. It decays as $1/R^2$, which is enough to compensate for the factor of R that arises in the numerator from the $|dz|$. If, however, we wished to evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} dx$$

then we do need the lemma, since a straightforward estimate will not give the required decay in this case. Here we must consider the complex integral

$$\int_\gamma \frac{ze^{iz}}{z^2+1}dz,$$

around the contour made up of Γ_R and the straight-line segment along the real axis from $-R$ to R. The integrand has poles at $z = \pm i$, and so the Residue theorem gives the result as

$$\int_\gamma \frac{ze^{iz}}{z^2+1}dz = 2\pi i \mathrm{Res}(ze^{iz}/(z^2+1); i) = 2\pi i \lim_{z\to i}(z-i)\frac{ze^{iz}}{z^2+1} = \frac{\pi i}{e}.$$

Thus,

$$\int_{-R}^{R} \frac{xe^{ix}}{x^2+1}\,dx + \int_{\Gamma_R} \frac{ze^{iz}}{z^2+1}dz = \frac{\pi i}{e}.$$

Since the function $z/(z^2+1)$ goes to zero uniformly on Γ_R as $R \to \infty$, Jordan's lemma 9.7 applies, and gives

$$\left|\int_{\Gamma_R} \frac{ze^{iz}}{z^2+1}dz\right| \to 0 \quad \text{as } R \to \infty.$$

Thus,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{xe^{ix}}{x^2+1}\,dx = \frac{\pi i}{e},$$

and taking the imaginary part gives the result we seek,

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx = \frac{\pi}{e}.$$

(Note that the real part gives

$$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+1}\,dx = 0,$$

a result we could have deduced by symmetry considerations.)

It is important to remember when the "$k < 0$" generalization of Jordan's lemma is required, because then the contour of integration to be used changes, and the residues must be taken from within the appropriate contour. The following example (taken from Ablowitz & Fokas [1]) shows such a case:

Example 9.8 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x \sin ax \cos bx}{x^2 + c^2} dx, \quad c > 0,\ a, b \in \mathbb{R}.$$

We first note that the trigonometric identity

$$\sin ax \cos bx = \frac{1}{2}(\sin(a-b)x + \sin(a+b)x)$$

suggests considering the complex integrals

$$J_1 = \int_{\gamma_1} \frac{ze^{i(a-b)z}}{z^2 + c^2} dz \quad \text{and} \quad J_2 = \int_{\gamma_2} \frac{ze^{i(a+b)z}}{z^2 + c^2} dz,$$

around appropriate contours γ, since the integrands here can be summed to give the result we want along the real axis. Suppose now for definiteness that $0 < a < b$ (leaving other cases for you to consider).

The contours of integration for each of J_1 and J_2 will be taken as large semicircular contours; and we require Jordan's lemma because a straightforward estimate of these integrals will not give convergence to zero on the circular portion of the contour. We take the function $f(z) = z/(z^2+c^2)$ in the statement of the lemma 9.7. The application of the lemma follows straightforwardly for J_2, since $k = a + b > 0$; and therefore taking $\gamma_2 = \Gamma_R^+ \cup \{[-R, R]\}$ and letting $R \to \infty$, we obtain

$$I_2 := \int_{-\infty}^{\infty} \frac{xe^{i(a+b)x}}{x^2 + c^2} dx = 2\pi i \text{Res}(ze^{i(a+b)z}/(z^2 + c^2); z = ic),$$

since there is just one pole of the integrand at $z = ic$ within the contour γ_2.

For J_1 the quantity $k = (a - b) < 0$, giving exponential divergence of the quantity e^{ikz} on the contour Γ_R^+ in the upper half plane as $R \to \infty$. However (as already noted) returning to the proof of the lemma we see that with $k < 0$, if we instead consider the contour Γ_R^- in the lower half plane, then the integral along this semicircle **will** go to zero as $R \to \infty$. Hence here we take our contour of integration $\gamma_1 = \Gamma_R^- \cup \{[R, -R]\}$ (note that the requirement for the contour to be positively oriented gives the line integral in the opposite sense). Letting $R \to \infty$ then, and applying the variant of Jordan's lemma, we obtain

$$-I_1 \quad := \quad \int_{\infty}^{-\infty} \frac{xe^{i(a-b)x}}{x^2 + c^2} dx = 2\pi i \text{Res}(ze^{i(a-b)z}/(z^2 + c^2); z = -ic),$$

since the only pole of the integrand inside γ_1 is $z = -ic$. Putting these results together then, the quantity we require is

$$\begin{aligned} I &= \frac{1}{2}\Im(I_1 + I_2) \\ &= \frac{2\pi}{2}\Re\left[-\mathrm{Res}\left(\frac{ze^{i(a-b)z}}{z^2+c^2}; z = -ic\right) + \mathrm{Res}\left(\frac{ze^{i(a+b)z}}{z^2+c^2}; z = ic\right)\right] \\ &= \pi\left(-\frac{e^{c(a-b)}}{2} + \frac{e^{-c(a+b)}}{2}\right) \\ &= -\pi e^{-bc}\sinh(ac). \end{aligned}$$

9.5 More complicated integration contours

There are many integrals we wish to evaluate for which a straightforward choice of integration contour (such as the unit circle, or the large semicircle in either lower or upper half-plane) will not give the result we need, even with an imaginative choice of integrand $f(z)$. In such cases we can sometimes be more creative with the choice of contour and devise a complex contour integral that can give the desired result; and we now consider several such examples.

9.5.1 Avoiding a singularity on the integration contour by indentation

If we wish to use residue calculus to evaluate improper integrals such as

$$I = \int_0^\infty \frac{\sin x}{x} dx,$$

then it can happen that the function we need to integrate has a singularity somewhere on the contour of integration. For the example above we might think to try the function $f(z) = \sin z/z$ integrated around a semicircular contour (this function has only a removable singularity at the origin, so we can define a convergent Taylor series expansion there, and the function is really analytic); but as we saw above with the cosine function, the sine function does not decay on the circular arc portion Γ_R of such a contour (whether in the upper or the lower half plane), but in fact grows exponentially as $R \to \infty$. So, we have to try integrating $f(z) = e^{iz}/z$, which decays exponentially on Γ_R as $R \to \infty$, and whose imaginary part on the real axis, $z = x \in \mathbb{R}$, is exactly the integrand we require. However, this function has a pole at $z = 0$, and so we cannot take a contour of integration that passes through the origin. The way around this difficulty is to use the basic semicircular contour, but make a small semicircular indentation (of vanishingly small radius $0 < \epsilon \ll 1$) around the pole at the origin. For such a contour γ we have, with $f(z) = e^{iz}/z$,

$$\int_\gamma f(z)\,dz = \int_{-R}^{-\epsilon} f(x)\,dx - \int_{\Gamma_\epsilon} f(z)\,dz + \int_\epsilon^R f(x)\,dx + \int_{\Gamma_R} f(z)\,dz = 0,$$

by Cauchy's theorem, where Γ_ϵ is the semicircular contour of radius ϵ in the upper half-plane. The integrals along the straight-line portions of this contour will give the result we seek, since in the limits $\epsilon \to 0$, $R \to \infty$ we have

$$\lim_{\epsilon\to 0, R\to\infty} \int_\epsilon^R f(z)\,dz = \int_0^\infty \frac{e^{iz}}{z} dz = \int_0^\infty \frac{\cos x + i\sin x}{x} dx,$$

and

$$\lim_{\epsilon\to 0, R\to\infty} \int_{-R}^{-\epsilon} f(z)\,dz = \int_{-\infty}^0 \frac{e^{iz}}{z} dz = \int_\infty^0 \frac{e^{-ix}}{x} dx = \int_0^\infty \frac{-\cos x + i\sin x}{x} dx,$$

so that

$$\lim_{\epsilon\to 0, R\to\infty} \left(\int_\epsilon^R + \int_{-R}^{-\epsilon} \right) \frac{e^{iz}}{z} dz = 2i \int_0^\infty \frac{\sin x}{x} dx.$$

For the integral along the small semicircle we have $z = \epsilon e^{i\theta}$, so that

$$\int_{\Gamma_\epsilon} \frac{e^{iz}}{z} dz = \int_0^\pi \frac{e^{i\epsilon(\cos\theta + i\sin\theta)}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\,d\theta \to \pi i \quad \text{as } \epsilon \to 0.$$

Finally, for the integral along the semicircular contour Γ_R^+ Jordan's lemma applies, giving

$$\int_{\Gamma_R^+} \frac{e^{iz}}{z} dz \to 0 \quad \text{as } R \to \infty.$$

Putting all this together, we have

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

The above uses a special case of a general result when we indent a contour of integration along a (vanishingly) small circular arc to avoid a singularity in the integrand.

Lemma 9.9 *Let f be analytic in some $B'(a;r)$ $(r > 0)$, with a simple pole of residue b at $z = a$. Then with $\Gamma_\epsilon = \{z : z = a + \epsilon e^{i\theta}, \theta_1 \le \theta \le \theta_2\}$, we have*

$$\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} f(z) dz = ib(\theta_2 - \theta_1).$$

Proof If f has a simple pole then by Laurent's theorem 8.1, on $B'(a;r)$ it has the form

$$f(z) = \frac{b}{z-a} + g(z), \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where g is analytic on $B(a;r)$ (the regular part of the Laurent expansion of f converges uniformly on $B(a;r)$). Thus, $|g|$ is certainly bounded on the compact set represented by Γ_ϵ: $|g| \le M$ say, some $M > 0$. Then, using the definition of Γ_ϵ we have

$$\begin{aligned} \int_{\Gamma_\epsilon} f(z)\, dz &= b \int_{\theta_1}^{\theta_2} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} d\theta + \int_{\theta_1}^{\theta_2} g(z) i\epsilon e^{i\theta}\, d\theta \\ &= ib(\theta_2 - \theta_1) + \int_{\theta_1}^{\theta_2} g(z) i\epsilon e^{i\theta}\, d\theta. \end{aligned}$$

Using the bound on g we have

$$\left| \int_{\theta_1}^{\theta_2} g(z) i\epsilon e^{i\theta}\, d\theta \right| \le \epsilon \int_{\theta_1}^{\theta_2} |g(z)| d\theta \le \epsilon M(\theta_2 - \theta_1) \to 0 \quad \text{as } \epsilon \to 0.$$

Note that this result does not generalize to the case of a stronger singularity, such as a pole of higher order at $z = a$, as a glance at the proof easily shows. However, sometimes we may be faced with an integral that appears to require this generalization, such as the following:

$$\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} dx.$$

The real integrand has only a removable singularity at the origin (consider the Taylor expansion of $x - \sin x$ about $x = 0$), but we know that we cannot consider the integral of $(z - \sin z)/z^3$ around the large semicircle as this grows exponentially on Γ_R as $R \to \infty$. The preceding results suggest that we need to consider the real part of the complex integrand $f_1(z) = (z + ie^{iz})/z^3$ which, along the real axis $z = x \in \mathbb{R}$, gives the function we want. However, the Laurent expansion of this function about $z = 0$ is

$$\begin{aligned} f_1(z) &= \frac{1}{z^3}\left(z + i\left(1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \cdots\right)\right) \\ &= \frac{i}{z^3}\left(1 - \frac{z^2}{2!} - \frac{iz^3}{3!} + \cdots\right), \end{aligned}$$

which has a triple pole at $z = 0$, so a simple indentation of the contour as in the lemma above will not work. We need to find some other function that also has $(x - \sin x)/x^3$ as its real part, but that has at most a simple pole at the origin, and such a function is

$$f(z) = \frac{z + ie^{iz} - i}{z^3},$$

with Laurent expansion (from f_1 above)

$$\begin{aligned} f(z) &= \frac{i}{z^3}\left(1 - \frac{z^2}{2!} - \frac{iz^3}{3!} + \cdots\right) - \frac{i}{z^3} \\ &= -\frac{i}{z}\left(\frac{1}{2!} + \frac{iz}{3!} + \cdots\right) \end{aligned}$$

(thus we have just a simple pole of residue $-i/2$). We can integrate this function around a large semicircle in the upper half-plane, with an indentation at the origin. The contribution along the portion Γ_R^+ will go to zero as $R \to \infty$ (Jordan's lemma and elementary bounds), and lemma 9.9 applies to the small indentation, giving

$$\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} f(z)\, dz \to i\pi \frac{(-i)}{2} = \frac{\pi}{2}.$$

Since $f(z)$ has no singularities within the integration contour chosen, as $\epsilon \to 0$, $R \to \infty$ we obtain (taking care to observe the sense of integration on each portion of the contour)

$$\frac{\pi}{2} = \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right)\left(\frac{x - \sin x}{x^3} + \frac{i(\cos x - 1)}{x^3}\right) dx.$$

Taking the real part gives the result we want (the imaginary part tells us that the integral, interpreted in the principal-value sense, of the odd function $(\cos x - 1)/x^3$ over the whole real line, is zero).

9.5.2 Integrals around "sector" contours

In the preceding examples we have either been integrating functions along the whole real line or, when seeking the integral over part of the real line (the positive real axis), we have only considered functions such that the integral along the negative real axis is the same as that along the positive real axis, so that the results double up. We now consider a couple of examples where this is not the case, but where taking an integral of an appropriate function around a sector contour (possibly indented) can give us the result we want.

Example 9.10 *Evaluate the improper integral*

$$\int_0^\infty \cos x^2 \, dx.$$

We recognize the integrand as being the real part of $f(z) = e^{iz^2}$ on the positive real axis, but if we use the usual semicircular contour then on $\Gamma_R^\pm$ we have

$$|e^{iz^2}| = e^{-R^2 \sin 2\theta},$$

which does not decay exponentially over the whole contour for either choice Γ_R^+ or Γ_R^-. We need to derive a variant of Jordan's lemma 9.7 appropriate to this integrand. Note that Jordan's inequality (81) gives

$$\frac{2}{\pi} \leq \frac{\sin 2\theta}{2\theta} \leq 1 \quad \text{for } \theta \in (0, \pi/4],$$

suggesting that we restrict the arc to a quarter of the usual,

$$\Gamma_{R,1/4} = \{z : z = Re^{i\theta},\, 0 \leq \theta \leq \pi/4\},$$

since on such an arc we have

$$\begin{aligned} \left| \int_{\Gamma_{R,1/4}} f(z)\, dz \right| &\leq \int_0^{\pi/4} e^{-R^2 \sin 2\theta} R \, d\theta \leq R \int_0^{\pi/4} e^{-4R^2\theta/\pi} \, d\theta = \frac{\pi}{4R}(1 - e^{-R^2}) \\ &\to 0 \text{ as } R \to \infty. \end{aligned}$$

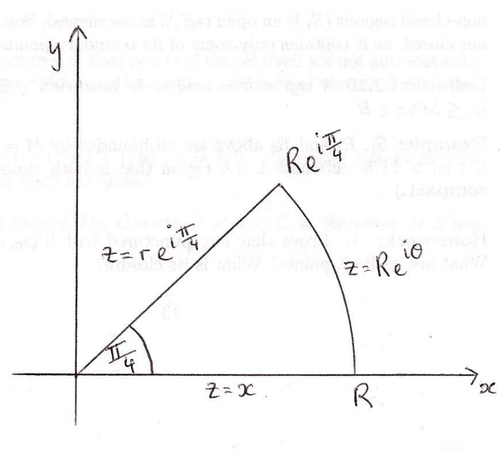

Figure 6: Sector contour of angle $\pi/4$ used in example 9.10.

Moreover, on the ray $z = re^{i\pi/4}$ we have $f(z) = e^{-r^2}$, which we can integrate from $r = 0$ to ∞. Thus taking the contour γ defined by

$$\gamma = \{[0, R]\} \cup \Gamma_{R,1/4} \cup \{z : z = re^{i\pi/4}, 0 \le r \le R\},$$

(see figure 6) we have $f(z)$ analytic inside γ and so by Cauchy's theorem,

$$\begin{aligned} 0 = \lim_{R\to\infty} \int_\gamma f(z)\, dz &= \int_0^\infty (\cos x^2 + i \sin x^2)\, dx - \int_0^\infty e^{-r^2} e^{i\pi/4}\, dr \\ \Rightarrow \quad \int_0^\infty (\cos x^2 + i \sin x^2)\, dx &= \int_0^\infty e^{-r^2} e^{i\pi/4}\, dr = \frac{(1+i)\sqrt{\pi}}{2\sqrt{2}}. \end{aligned}$$

Taking the real part gives the integral required; the imaginary part gives an equivalent result for the integral of $\sin x^2$.

Another example where a sector contour can be used is the following (from Ablowitz & Fokas, p.230): Evaluate

$$I = \int_0^\infty \frac{dx}{x^3 + a^3}, \quad a > 0.$$

In this case, the fact that the integrand is not an even function of x prevents us from using the usual semicircular contour, since the integral of $f(z) = 1/(z^3+a^3)$ along the negative real axis does not give a multiple of I. However, if we consider rays in the complex plane again, $z = re^{i\alpha}$ for fixed α and

$r \in \mathbb{R}^+$, then on such a ray $f(z) = 1/(r^3 e^{3i\alpha} + a^3)$; and we see that if $\alpha = 2\pi/3$ then we get the desired integrand, $f(z = 1/(r^3 + a^3)$. Thus, if we take the integral of $f(z)$ around a large sector of angle $2\pi/3$ we will get contributions proportional to I from the two straight sides, a vanishing contribution from the curved portion as $R \to \infty$; and the total value of the integral will be given by Cauchy's residue theorem, since there is a simple pole of the integrand at $z = ae^{i\pi/3}$ inside the contour. Denoting the sector contour by γ we have on the one hand, using the residue theorem,

$$\begin{aligned}
\int_\gamma \frac{dz}{z^3 + a^3} &= 2\pi i \mathrm{Res}(f(z); ae^{i\pi/3}) \\
&= 2\pi i \lim_{z \to ae^{i\pi/3}} \frac{z - ae^{i\pi/3}}{z^3 + a^3} \\
&= \left. \frac{2\pi i}{3z^2} \right|_{z = ae^{i\pi/3}} \\
&= \frac{2\pi i e^{-2\pi i/3}}{3a^2} \\
&= \frac{2\pi e^{-i\pi/6}}{3a^2},
\end{aligned}$$

where we used l'Hôpital's rule between the 2nd and 3rd lines to evaluate the limit. On the other hand, splitting the contour into its constitutive parts we have

$$\lim_{R \to \infty} \int_\gamma f(z)\, dz = \int_0^\infty \frac{dx}{x^3 + a^3} - e^{2\pi i/3} \int_0^\infty \frac{dr}{r^3 + a^3} = (1 - e^{2\pi i/3}) I.$$

Thus,

$$I = \frac{2\pi e^{-i\pi/6}}{3a^2(1 - e^{2\pi i/3})} = \frac{-2\pi e^{-i\pi/6}}{3a^2 e^{i\pi/3}(e^{i\pi/3} - e^{-i\pi/3})} = \frac{2\pi i}{3a^2.2i \sin \pi/3} = \frac{2\pi}{3a^2\sqrt{3}}.$$

9.5.3 Integrals of functions with branch-points

So far we have considered only integrals of functions that have isolated singularities in the complex plane. However, as long as we are careful to define analytic branches of "multifunctions" with branch-points, and as long as we work in the appropriate cut plane and do not try to integrate across branch-cuts, we can take contour integrals of functions with branch points also. As usual, rather than prove general results, it is best to illustrate by examples.

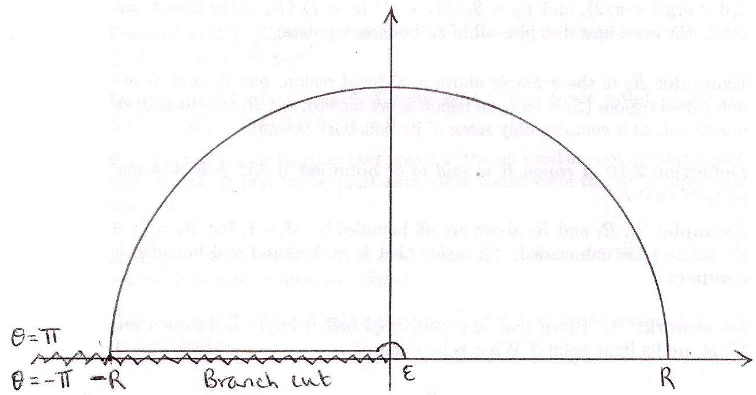

Figure 7: A semicircular contour with branch-cut along the negative real axis, used in example 9.11.

Example 9.11 *Evaluate*

$$\int_0^\infty \frac{\log x}{1+x^2}dx.$$

We consider integrating the complex function $f(z) = \log z/(1+z^2)$ around a suitable contour. We define an analytic branch of this function by cutting the z-plane along the negative real axis, taking the analytic branch of the logarithm given by $\log z = \log |z|+i\theta$, where θ is an argument of z ($z = |z|e^{i\theta}$) and $-\pi < \theta \leq \pi$. Then, as long as we stay on one side of the cut, $\theta = \pi^-$, we can use the same indented semicircular contour γ as before (we need to indent along an ϵ-semicircle about $z = 0$ to avoid the singularity there); see figure 7. With the indented semicircle in the upper half-plane we have a single pole of the integrand inside the contour at $z = i$, so that

$$\int_\gamma \frac{\log z}{1+z^2}dz = 2\pi i \mathrm{Res}(f(z); i).$$

The contour integral has four contributions. Along Γ_R^+ we have $z = Re^{i\theta}$, $0 \leq \theta < \pi$, so that

$$\begin{aligned}
\left|\int_{\Gamma_R^+} f(z)\,dz\right| &= \left|\int_0^\pi \frac{\log R + i\theta}{1+R^2e^{2i\theta}} iRe^{i\theta}\,d\theta\right| \\
&\leq \frac{R}{R^2-1}\int_0^\pi |\log R + i\theta| d\theta \leq \frac{\pi R\log R}{R^2-1} + \frac{\pi^2 R}{R^2-1} \to 0 \quad \text{as } R \to \infty.
\end{aligned}$$

Along the ϵ-semicircle we need another direct estimate of the integral. With $z = \epsilon e^{i\theta}$

$$\begin{aligned}\left|\int_{\Gamma_\epsilon} f(z)\,dz\right| &= \left|\int_0^\pi \frac{\log\epsilon + i\theta}{1+\epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta}\,d\theta\right| \\ &\le \frac{\epsilon(|\log\epsilon| + \pi)\pi}{1-\epsilon^2} \to 0 \quad \text{as } \epsilon \to 0.\end{aligned}$$

Along the positive real axis we have $z = x \in \mathbb{R}^+$, and as $\epsilon \to 0$ and $R \to \infty$ this integral converges to the one we want. Along the negative real axis portion of γ we have, since we are on the top side of the cut, $z = re^{i\pi}$ ($r = |z|$); $dz = e^{i\pi}dr = -dr$ and $\log z = \log r + i\pi$, and this portion of the integral is, as $\epsilon \to 0$, $R \to \infty$,

$$\int_\infty^0 \frac{\log r + i\pi}{1+r^2}(-dr) = \int_0^\infty \frac{\log r}{1+r^2}dr + i\pi\int_0^\infty \frac{dr}{1+r^2}.$$

Again, the first integral here is equal to the one we seek, and putting all results together we have, letting $\epsilon \to 0$ and $R \to \infty$,

$$2\int_0^\infty \frac{\log x}{1+x^2}dx = 2\pi i \mathrm{Res}(f(z); i) - i\pi\int_0^\infty \frac{dr}{1+r^2};$$

so we can see we will get two real integrals for the price of one here. Since f has only a simple pole at $z = i$ we have

$$\mathrm{Res}(f(z); i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{\log z}{z^2+1} = \frac{\log(i)}{2i} = \frac{\pi}{4}.$$

Thus we find

$$\int_0^\infty \frac{\log x}{1+x^2}dx = 0, \quad \text{and} \quad \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Example 9.12 *Show that*

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \pi\, cosec\pi a \quad (0 < a < 1).$$

The integrand here is another multifunction when considered in the complex plane, so we must again cut the plane appropriately and define an analytic branch of the multifunction in the cut plane. This time we make a cut along

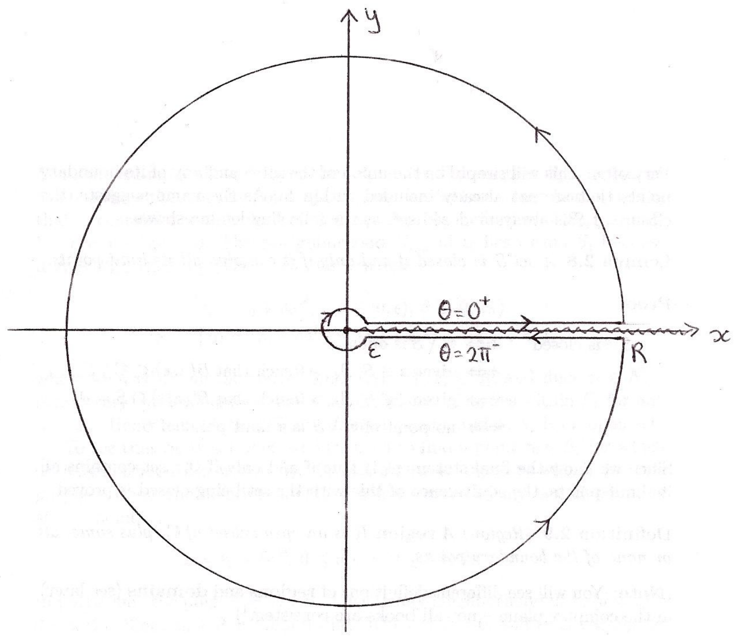

Figure 8: “Keyhole” contour used in example 9.12.

the positive real axis, so that we represent z in the cut plane by $z = |z|e^{i\theta}$, $0 \leq \theta < 2\pi$ and define the analytic branch of z^{a-1} by

$$z^{a-1} = |z|^{a-1}e^{i(a-1)\theta}.$$

The contour used in the previous example does not help us here, since the integral along the negative real axis won't reduce to something related to the integral we want (check this). So, we choose a contour that includes straight-line integrals along both sides of the cut, with a small circular indentation (a “keyhole”) to avoid the branch-point itself (figure 8). The contour is closed off by a large circular arc of radius $R \gg 1$ that does not cross the cut. This contour is well-known, and is often called a keyhole contour. Within the contour the chosen branch of the function is analytic except for a simple pole at $z = -1$; and denoting by γ the entire contour we have

$$\int_\gamma f(z)\, dz = 2\pi i \mathrm{Res}(f(z); z = -1).$$

On the outer large circular portion of γ we have $z = Re^{i\theta}$, $0 \le \theta < 2\pi$, and so

$$|f(z)| = \left| \frac{R^{a-1}e^{i(a-1)\theta}}{1 + Re^{i\theta}} \right| \le \frac{R^{a-1}}{R-1}.$$

Thus, using also $dz = iRe^{i\theta}d\theta$ on Γ_R,

$$\left| \int_{\Gamma_R} f(z)\, dz \right| \le \int_0^{2\pi} \frac{R^{a-1}R}{R-1} d\theta \to 0 \quad \text{as } R \to \infty \text{ (since } a < 1).$$

The integral along the upper side of the cut converges trivially to the desired integral as $\epsilon \to 0$, $R \to \infty$. Along the bottom side of the cut the contribution to $\int_\gamma$ is such that $z = re^{2\pi i}$ in the integrand, with r going from ∞ to 0 to preserve the same sense of integration. Thus we have $dz = dr$, and, if we call this portion of the curve γ^- then in the limit $\epsilon \to 0$, $R \to \infty$

$$\int_{\gamma^-} f(z)\, dz = \int_\infty^0 \frac{r^{a-1}e^{2\pi i(a-1)}}{1+r}\, dr = -e^{2\pi i(a-1)} \int_0^\infty \frac{r^{a-1}}{1+r}\, dr$$

which is a (complex) multiple of the integral we want. Finally, we also need to estimate the integral around the indented ϵ-circle Γ_ϵ and show that this goes to zero as $\epsilon \to 0$. The same estimate used for the large circle Γ_R gives

$$\left| \int_{\Gamma_\epsilon} f(z)\, dz \right| \le \int_0^{2\pi} \frac{\epsilon^{a-1}\epsilon}{1-\epsilon} d\theta \to 0 \quad \text{as } \epsilon \to 0 \text{ (since } a > 0).$$

Putting the above results together we have

$$\begin{aligned}
(1 - e^{2\pi i(a-1)}) \int_0^\infty \frac{x^{a-1}}{1+x}\, dx &= 2\pi i \operatorname{Res}(f(z); z = -1) \\
&= 2\pi i\, z^{a-1}\big|_{z=-1} \\
&= 2\pi i e^{i\pi(a-1)},
\end{aligned}$$

taking care to use the analytic branch of the multifunction as defined above. Denoting by I the real integral we seek, we then have

$$\begin{aligned}
-2iIe^{i\pi(a-1)} \sin \pi(a-1) &= 2\pi i e^{i\pi(a-1)} \\
\Rightarrow \quad I &= \frac{\pi}{\sin \pi a}.
\end{aligned}$$

Remark When calculating integrals such as this, involving integrating a multifunction in a cut plane, it is important to ensure that you remain consistent in your choice of analytic branch throughout. In example 9.12 just solved, for example, we would have obtained an incorrect answer had we worked with the branch

$$z^{a-1} = |z|^{a-1} e^{i(a-1)\theta}, \quad -\pi < \theta < \pi.$$

This does define an analytic branch, but only in the plane cut along the **negative** real axis – which is inconsistent with the contour we have chosen.

9.5.4 Integrals around rectangular contours

For certain classes of integrals involving functions based on exponentials, it can be useful to take a rectangular contour of integration, one side of the rectangle being (usually in a limiting sense) the integral we want, and the opposite side giving a related integral. A rectangular contour can be useful for two reasons: firstly, we may wish to integrate a function that has an infinite number of singularities, e.g. along the imaginary axis. A rectangular contour of finite width in the y-direction will enclose only a finite number of the singularities, giving a finite sum of residues, where the limiting semicircular contour of figure 4 (or its reflection in the x-axis) leads to an infinite sum of residues. Secondly, the integrand may not decay over the whole of the circular portion of the contour (and a sector may not help either, unless the integral along the second sector arc gives an integral related to the one we want, or one that we can easily evaluate).

Example 9.13 *Evaluate*

$$I = \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\, dx, \quad 0 < p < 1.$$

As usual, we consider the limit in which a large but finite sized contour goes to infinity in some sense. There is no convenient choice of semicircle (or sector) contour for this example – consideration of integrands such as $e^{pz}/(1+e^z)$ integrated along the real axis, or $e^{-ipz}/(1+e^{-iz})$ integrand along the imaginary axis, cannot be closed off with a semicircle on which the integrand decays uniformly. Moreover, in each case, the integrand has an infinite set of singular points (along the imaginary or real axis), which would all be enclosed in the limit that a semicircular integration contour goes to infinity.

We instead take a rectangle, with one side along the real axis from $-R$ to R (along $y=0$), and the parallel side along $y=Y$. The two shorter sides along $x=\pm R$ complete the rectangle. We need to make a convenient choice of Y that allows us to evaluate the real integral we want. We choose complex integrand $f(z)=e^{pz}/(1+e^z)$, since this gives the correct integral along the real axis. On $y=Y$ we have

$$f(z)=\frac{e^{p(x+iY)}}{1+e^{x+iY}}=\frac{e^{px}e^{ipY}}{1+e^xe^{iY}},$$

and we see that if we choose $Y=2\pi$ then we get a multiple of the integral we want. On the short sides of the rectangle, Γ_r and Γ_l say, we have $z=\pm R+iy$, $0\le y\le 2\pi$. On Γ_r,

$$\begin{aligned}\left|\int_{\Gamma_r} f(z)\,dz\right| &= \left|\int_0^Y \frac{e^{pR}e^{iy}}{1+e^Re^{iy}}idy\right| \le \int_0^Y \frac{e^{pR}}{e^R-1}\,dy \le \frac{Ye^{pR}}{e^R-1}\\ &\to 0 \quad \text{as } R\to\infty,\end{aligned}$$

since $p<1$. On Γ_l we have

$$\begin{aligned}\left|\int_{\Gamma_l} f(z)\,dz\right| &= \left|\int_Y^0 \frac{e^{-pR}e^{iy}}{1+e^{-R}e^{iy}}idy\right| \le \int_0^Y \frac{e^{-pR}}{1-e^{-R}}\,dy \le \frac{Ye^{-pR}}{1-e^{-R}}\\ &\to 0 \quad \text{as } R\to\infty,\end{aligned}$$

since $p>0$. By the Residue theorem then, as $R\to\infty$ we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{e^{px}}{1+e^x}\,dx+\int_{\infty}^{-\infty}\frac{e^{2\pi ip}e^{px}}{1+e^x}\,dx = 2\pi i\mathrm{Res}(f(z),i\pi)\\ \Rightarrow\quad &(1-e^{2\pi ip})I = 2\pi i\lim_{z\to i\pi}(z-i\pi)\frac{e^{pz}}{1+e^z}=2\pi i\frac{e^{i\pi p}}{e^{i\pi}}=-2\pi ie^{i\pi p}.\end{aligned}$$

Thus,

$$I=\frac{2\pi ie^{i\pi p}}{e^{i\pi p}\big(e^{i\pi p}-e^{-i\pi p}\big)}=\frac{\pi}{\sin\pi p}.$$

We conclude by considering a more complicated variant.

Example 9.14 *Evaluate*

$$I=\int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^x}\,dx,\quad 0<p,q<1.$$

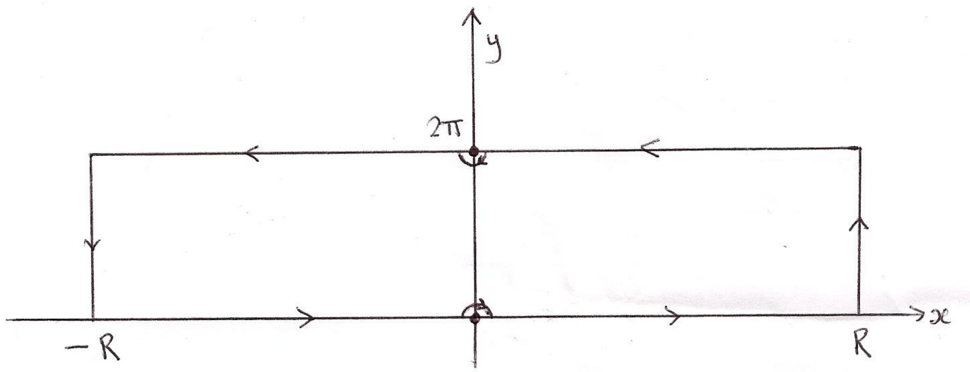

Figure 9: Rectangle with indentations to exclude simple poles from the interior, used in example 9.14.

Choosing $f(z) = (e^{pz} - e^{qz})/(1-e^z)$, there is no convenient choice of rectangle size Y that will give a multiple of the integrand we want on $\Im(z) = Y$. However, we could evaluate two contributions separately, similar to what was done above. The only slight complication is that each of the integrands

$$f_p(z) = \frac{e^{pz}}{1-e^z}, \quad f_q(z) = \frac{e^{qz}}{1-e^z},$$

have simple poles on their integration contours, at $z = 0$ and $z = 2\pi i$, which we must indent around. Note that if we work with these integrands, then we are evaluating principal-value integrals, since the singularity in the integrand is no longer removable. Writing

$$I_p = \int_{-\infty}^{\infty} \frac{e^{px}}{1-e^x}\,dx, \quad I_q = \int_{-\infty}^{\infty} \frac{e^{qx}}{1-e^x}\,dx,$$

we have $I = I_p - I_q$. We integrate around a closed rectangular contour γ, which we take to be the same as in the previous example, except for two indentations of radius ϵ that exclude the simple poles at $z = 0$ and $z = 2\pi i$ from the contour interior (figure 9). Neither integrand is then singular inside γ, so Cauchy's theorem applies to each of f_p and f_q. The integral along the real axis will give the result we want, in the limit that the indentation radius $\epsilon \to 0$ and the rectangle size $R \to \infty$. The integral along $\Im(z) = R$ will again give a multiple of this integral. The contributions from the contour indentations will come from Lemma 9.9, noting that for each case, the indentation round the pole is taken in the opposite sense to that in the lemma statement (so we get a change of sign). For the integrand f_p we thus obtain

$$I_p - \int_{-\infty}^{\infty} \frac{e^{2\pi i p} e^{px}}{1 - e^x}\, dx - i\pi \mathrm{Res}(f_p(z), 0) - i\pi \mathrm{Res}(f_p(z), 2\pi i) = 0.$$

For the residues we have

$$\mathrm{Res}(f_p(z), 0) = \lim_{z \to 0} \frac{z e^{pz}}{1 - e^z} = -1,$$
$$\mathrm{Res}(f_p(z), 2\pi i) = \lim_{z \to 2\pi i} \frac{(z - 2\pi i) e^{pz}}{1 - e^z} = -e^{2\pi i p},$$

and so

$$I_p(1 - e^{2\pi i p}) = -i\pi(1 + e^{2\pi i p})$$
$$\Rightarrow \quad I_p = i\pi \frac{e^{2\pi i p} + 1}{e^{2\pi i p} - 1} = i\pi \frac{e^{\pi i p} + e^{-\pi i p}}{e^{\pi i p} - e^{-\pi i p}} = \pi \cot \pi p.$$

Hence finally, since the result for I_q follows exactly similarly,

$$I = \pi(\cot \pi p - \cot \pi q).$$

Homework (1) What is wrong with the following argument: "Let $I = \int_0^\infty (1 + x^4)^{-1}\, dx$. Put $x = iy$, then

$$I = \int_0^\infty (1 + y^4)^{-1}\, dy = iI.$$

Hence, $I = 0$." How do we obtain the correct value of the integral $(\pi/(2\sqrt{2}))$?

(2) Prove that

$$(i) \qquad \int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}(b - a), \quad (a, b > 0),$$
$$(ii) \qquad \int_0^\infty \frac{(\log x)^2}{1 + x^2} dx = \frac{\pi^3}{8}.$$

9.6 Principle of the argument, and Rouché's theorem

Residue calculus can also be applied to deduce results about the number of zeros and poles of a meromorphic function in a given region.

Theorem 9.15 *(Argument principle) Let f be meromorphic inside and on a simple closed contour C, with N zeros inside C, P poles inside C (where multiple zeros and poles are counted according to multiplicity), and no zeros or poles on C. Then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P = \frac{1}{2\pi}[\arg(f(z))]_C$$

(the last expression denotes the change in the argument of f as C is traversed once).

Proof Let z_i be a zero/pole of order n_i. Then

$$f(z) = (z - z_i)^{\pm n_i} g_i(z), \qquad (+ \text{ for zero, } - \text{ for pole})$$

where $g_i(z_i) \neq 0$ and $g_i(z)$ is analytic and nonzero in some neighborhood $B(z_i, \epsilon_i)$ of z_i (see lemma 7.12). Thus in such a neighborhood we can write

$$\frac{f'(z)}{f(z)} = \frac{\pm n_i}{(z - z_i)} + \phi_i(z)$$

where $\phi_i(z) = g_i'(z)/g_i(z)$ is analytic in $B(z_i, \epsilon_i)$, and in the region D' made up of the interior of the curve C, with each $B(z_i, \epsilon_i)$ removed, f'/f is analytic. Application of Cauchy's theorem 5.10 to the region D' then gives

$$\begin{aligned}
0 &= \frac{1}{2\pi i}\oint_{\partial D'} \frac{f'(z)}{f(z)}dz \\
&= \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz - \frac{1}{2\pi i}\sum_i \oint_{\gamma(z_i,\epsilon_i)} \frac{f'(z)}{f(z)}dz \\
&= \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz - \frac{1}{2\pi i}\sum_i \oint_{\gamma(z_i,\epsilon_i)} \frac{\pm n_i}{(z - z_i)} + \phi_i(z)dz \\
\Rightarrow \qquad & \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \sum_{zeros} n_i - \sum_{poles} n_j = N - P,
\end{aligned}$$

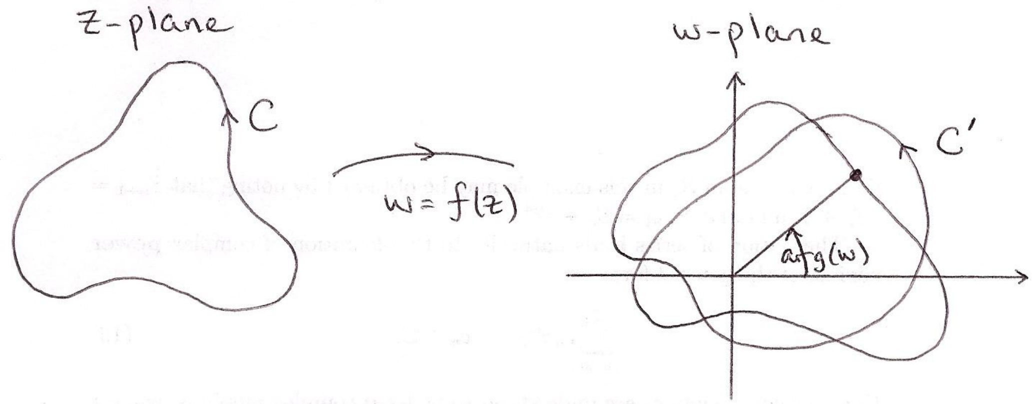

Figure 10: A geometrical interpretation of the Argument Principle.

where $\gamma(z_i, \epsilon_i)$ denotes the circle of center z_i and radius ϵ_i, and we used the residue theorem 9.4 in the last step. The quantities N, P are as defined in the theorem statement.

To demonstrate the final equality in the theorem, we parametrize the curve C so that

$$C = \{z(t) : t \in [a, b], z(a) = z(b)\}.$$

Then

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))}{f(z(t))}z'(t)dt = \frac{1}{2\pi i}[\log|f(z(t))| + i\arg(f(z(t)))]_{t=a}^b \\ &= \frac{1}{2\pi}[\arg(f(z))]_C\end{aligned}$$

(which branch of the logarithm we choose is immaterial for the theorem result, but for definiteness you can assume the principal branch).

The theorem has an interesting geometrical interpretation, illustrated in figure 10. The function $w = f(z)$ represents a mapping from the complex z-plane to the complex w-plane, under which the curve C maps to some image curve C' in the w-plane (see later notes on conformal mappings; Math 756). We then have $dw = f'(z)dz$, so that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C'} \frac{dw}{w} = \frac{1}{2\pi i}[\log(w)]_{C'} = \frac{1}{2\pi}[\arg(w)]_{C'}.$$

Remark This quantity $(1/2\pi)[\arg(w)]_{C'}$ is known as the **winding number** of the curve C' about the origin in the w-plane – the number of times that the closed curve C' encircles the origin.

Example 9.16 *Find the number of zeros of the function $f(z) = z^5+1$ within the first quadrant.*

We use the principle of the argument to determine the change in $\arg(f)$ as we traverse an appropriate contour. Since f is analytic on the first quadrant of the z-plane, we may take the contour C to be a large quarter-circle, made up of the three portions $C = C_1 \cup C_2 \cup C_3$,

$$\begin{aligned} C_1 &= \{z : z = x, \quad 0 \leq x \leq R\}, \\ C_2 &= \{z : z = Re^{i\theta}, \quad 0 \leq \theta \leq \pi/2\}, \\ C_3 &= \{z : z = iy, \quad 0 \leq y \leq R\}. \end{aligned}$$

In the limit that R becomes arbitrarily large the contour will enclose the whole of the first quadrant. The argument of f satisfies $\arg(f) = \phi$, where $\tan\phi = \Im(f)/\Re(f)$. Along C_1 $f = 1 + x^5 \in \mathbb{R}^+$, and $\arg(f) = 0$. Along C_2 $f = 1 + R^5e^{5i\theta} \approx R^5e^{5i\theta}$, and the argument of f therefore increases from 0 to $5\pi/2$ (in the limit $R \to \infty$) as θ increases from 0 to $\pi/2$. Finally, on C_3 $f = 1 + iy^5$, and for $y = R \gg 1$ we have $\arg(f) \approx 5\pi/2$ (it must vary continuously from C_2 to C_3), while as y decreases from $R \gg 1$ to 0, $\tan\phi = \Im(f)/\Re(f)$ decreases (continuously) from $+\infty$ to 0^+ and thus $\arg(f)$ decreases (continuously) from $5\pi/2$ to 2π.

Thus, the net change in $\arg(f)$ as we traverse C is 2π. Applying theorem 9.15 then, since f has no singularities within the first quadrant (so $P = 0$), the number of zeros it has, N, satisfies

$$N = \frac{1}{2\pi}[\arg(f)]_C = \frac{2\pi}{2\pi} = 1.$$

Theorem 9.17 *(Rouché) Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed contour C. If $|f| > |g|$ on C, then f and $f + g$ have the same number of zeros inside C.*

This theorem makes sense intuitively, since we can think of g as being a perturbation to f, which will in turn perturb its zeros. If the size of the perturbation is bounded on the contour C, then (we can show) the size of the perturbation is bounded inside C as well (this is not surprising, given

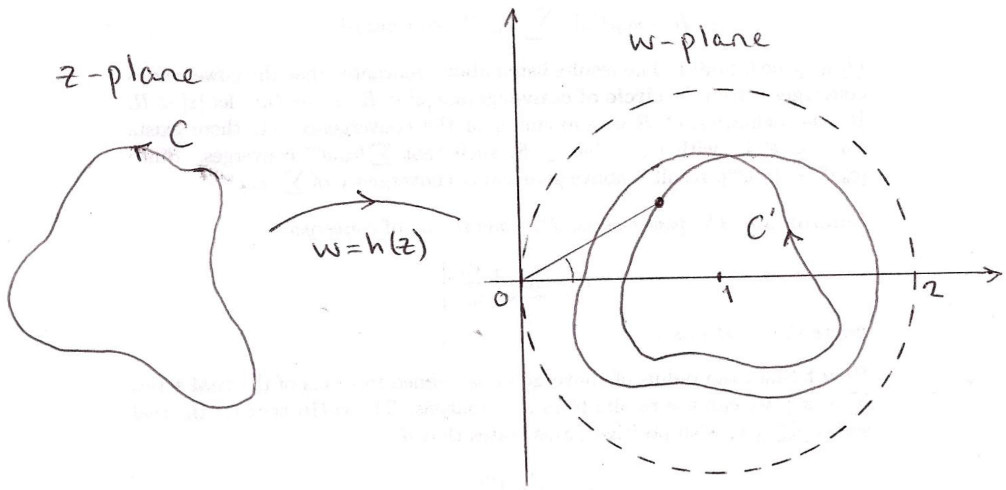

Figure 11: Illustration of the proof of Rouch'e's theorem.

the maximum modulus principle for analytic functions). Therefore, we have not perturbed the zeros of f too much by adding g.

Proof Since $|f| > |g| \geq 0$ on C it follows that $|f| > 0$ on C and thus $f \neq 0$ on C. Moreover, $f(z) + g(z) \neq 0$ on C. Let

$$h(z) = \frac{f(z) + g(z)}{f(z)},$$

then h is analytic and nonzero on C, and the Argument Principle theorem 9.15 may be applied to deduce that

$$\frac{1}{2\pi i} \oint_C \frac{h'(z)}{h(z)} dz = \frac{1}{2\pi} [\arg(w)]_{C'},$$

where $w = h(z)$, and C' is the image of the curve C under this transformation in the w-plane. However,

$$w = h(z) = 1 + \frac{g(z)}{f(z)}$$

so, since $|g| < |f|$ on C, we have $|w - 1| < 1$ on C', so that all points of C' lie within the circle of radius 1 centered at $w = 1$ (see figure 11). Thus, as we traverse the closed curve C' there is no net change in $\arg(w)$, because C' does not enclose the origin. It follows from the argument principle that $N_h - P_h = 0$, where N_h and P_h are the numbers of zeros and poles, respectively (counted according to multiplicity) of $h(z)$. Since f and g are analytic inside and on C, the poles of h coincide in location and multiplicity with the zeros of f, so that $P_h = N_f$. Also by analyticity of f, h is zero only where $f+g=0$, and we have $N_h = N_{f+g}$. Thus,

$$N_h - P_h = 0 \qquad \Rightarrow \qquad N_f = N_{f+g},$$

and the theorem is proved.

Example 9.18 *Show that $4z^2 = e^{iz}$ has a solution on the unit disc $|z| \leq 1$.*

Take the contour C to be $|z| = 1$, so that points on C are given by $z = e^{it}$, $t \in [0, 2\pi]$. Then, with $f(z) = 4z^2$ and $g(z) = -e^{iz}$, we have $|f| = 4$ on C, while

$$|g(z)|_C = |e^{i(\cos t + i \sin t)}| = e^{-\sin t} \leq e < |f(z)|_C.$$

Thus Rouché's theorem applies, and $f+g$ has the same number of zeros as f on the unit disc. Clearly, $f(z) = 4z^2$ has exactly 2 zeros (both at $z = 0$ but we count according to multiplicity). It follows that $4z^2 = e^{iz}$ in fact has 2 solutions on the unit disc.

Homework: Ablowitz & Fokas, problems for section 4.4, questions 3(b), 5(a).

10 Asymptotic expansions and Stokes lines

10.1 Asymptotic sequences and asymptotic expansions

Asymptotic expansions are about finding an approximation to some function that depends on a small (usually real) parameter, $0 < \epsilon \ll 1$. The function may be given explicitly, with ϵ-dependence embedded in the definition, or (as is more common in applications) the function may be a solution of a differential equation (ordinary or partial) that contains the small parameter. Typically the approach is to find some means of **expanding** the function as a sum, often in powers of the small parameter, to a certain number of terms, *e.g.*

$$\begin{aligned} f(\epsilon) &= f_0 + \epsilon f_1 + \cdots + \epsilon^N f_N + R_N(\epsilon) \\ &= S_N(\epsilon) + R_N(\epsilon), \end{aligned} \tag{84}$$

in order to achieve a desired level of accuracy. Here, $R_N(\epsilon)$ is some remainder term, which intuitively we would expect to be much smaller than the preceding term in the series. In most asymptotic expansions in ϵ of the above form, we in fact have $|R_N(\epsilon)| = O(\epsilon^{N+1})$ as $\epsilon \to 0$.[1] Loosely speaking, this means that $|R_N(\epsilon)|$ is bounded by some constant (positive) multiple of ϵ^{N+1} as $\epsilon \to 0$; more precisely, there exists some $M > 0$ and some $\epsilon_0 > 0$ (small) such that

$$|R_N(\epsilon)| \leq M\epsilon^{N+1} \quad \text{for } \epsilon < \epsilon_0.$$

Thus, for example, any polynomial in ϵ, $P(\epsilon) = \sum_{n=1}^{N} a_n \epsilon^n$, is of order ϵ as $\epsilon \to 0$. As well as this "order" notation, we also have the concept of **smaller order"**, denoted by "o" instead of "O". Here we write

$$f(\epsilon) = o(g(\epsilon)) \quad \text{as } \epsilon \to 0 \tag{85}$$

to signify that $f(\epsilon)$ is much smaller than $g(\epsilon)$ as $\epsilon \to 0$ – again, more precisely, we have

$$\lim_{\epsilon \to 0} \left| \frac{f(\epsilon)}{g(\epsilon)} \right| \to 0 \quad \text{as } \epsilon \to 0.$$

The relation (85) may be alternatively expressed as

$$f(\epsilon) \ll g(\epsilon) \quad \text{as } \epsilon \to 0.$$

[1] In words: $|R_N(\epsilon)|$ is of order ϵ^{N+1} as $\epsilon \to 0$.

Returning to our initial idea of approximating a function (see (84)), we say that S_N is an approximation to f valid, or correct, or accurate, to order ϵ^N as $\epsilon \to 0$ if

$$\lim_{\epsilon \to 0} \frac{f - S_N}{\epsilon^N} \equiv \lim_{\epsilon \to 0} \frac{R_N}{\epsilon^N} = 0.$$

Thus, if indeed $R_N = O(\epsilon^{N+1})$ as $\epsilon \to 0$, or even if $R_N = o(\epsilon^N)$ (a less strong condition) then S_N does approximate f to order ϵ^N.

While in many cases it is easy to write down the approximation to a function to a desired (quantifiable) level of accuracy, using our knowledge of convergent series expansions, there are many situations in which such notions are not fully adequate. For example, we may seek to approximate a function for which there is no closed-form expression – it may be an unknown solution to a differential equation, or be defined in terms of an integral. For example, we can say with confidence, using the terminology developed above, that

$$\cos \epsilon = 1 - \frac{\epsilon^2}{2!} + R, \quad \text{where} \quad R = O(\epsilon^4) \quad \text{as } \epsilon \to 0$$

(R has a convergent Taylor series expansion from which we can extract a factor of ϵ^4 and still obtain a bounded result as $\epsilon \to 0$). Thus, $1 - \epsilon^2/2!$ is an approximation to $\cos \epsilon$, valid to order ϵ^3. This is an easy case to analyze; but what about, e.g.,

$$f(\epsilon) = \int_0^\infty \frac{e^{-t}\, dt}{1 + \epsilon t} \quad \text{as } \epsilon \to 0? \tag{86}$$

We could, disregarding all rigor, attempt to expand the denominator in the integrand using the binomial theorem; and then (even worse!) interchange the summation and integration:

$$\begin{aligned} \int_0^\infty \frac{e^{-t}}{1+\epsilon t} &\overset{?}{\sim} \int_0^\infty \sum_{n=0}^\infty (-1)^n \epsilon^n t^n \, dt \\ &\overset{?}{\sim} \sum_0^\infty (-1)^n \epsilon^n \int_0^\infty e^{-t} t^n \, dt. \end{aligned}$$

Writing $I_n = \int_0^\infty e^{-t} t^n \, dt$, we easily derive the recurrence relation $I_n = nI_{n-1}$, so that $I_n = n! I_0 = n!$, and so

$$\int_0^\infty \frac{e^{-t}}{1+\epsilon t} \overset{?}{\sim} \sum_{n=0}^\infty (-1)^n n! \epsilon^n; \tag{87}$$

but this last series diverges for all $\epsilon > 0$. **However,** this last exercise is not futile, because it turns out that if one **truncates** the divergent infinite series in (87) at a finite number of terms, N, then for small values of ϵ the finite sum gives a good approximation to the integral, and the approximation gets better and better as $\epsilon \to 0$. It can be shown that

$$\int_0^\infty \frac{e^{-t}\,dt}{1+\epsilon t} = \sum_0^N (-1)^n n!\epsilon^n + O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0 \textbf{ for } N \textbf{ fixed}. \tag{88}$$

This result can be perhaps more convincingly arrived at by noting that, if we define

$$I_n(\epsilon) = \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^n}\,dt,$$

then a single integration by parts gives

$$I_n = \left[\frac{-e^{-t}}{(1+\epsilon t)^n}\right]_0^\infty - \epsilon n \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^{n+1}}\,dt = 1 - n\epsilon I_{n+1},$$

and thus

$$\begin{aligned} I_1 &= 1 - 1.\epsilon I_2 = 1 - 1.\epsilon(1 - 2\epsilon I_3) = 1 - \epsilon + 2!\epsilon^2(1 - 3\epsilon I_4) \\ &\vdots \\ &= 1 - \epsilon + 2!\epsilon^2 - 3!\epsilon^3 + \cdots + (-1)^N N!\epsilon^N + (-1)^{N+1}(N+1)!\epsilon^{N+1} I_{N+2}, \end{aligned}$$

where $I_{N+2} = O(1)$ as $\epsilon \to 0$ at fixed N (with limit 1, in fact).

On the other hand, in contrast to (88)

$$\cos\epsilon = \sum_{n=0}^N \frac{(-1)^n \epsilon^{2n}}{(2n)!} + O\left(\frac{\epsilon^{2N+2}}{(2N+2)!}\right) \quad \text{as } N \to \infty \text{ for } \epsilon \text{ fixed}$$

(though in fact this result also holds as $\epsilon \to 0$ with N fixed). The sum on the right-hand-side of (88) provides an **asymptotic expansion** of the integral on the left-hand side; and we write

$$f(\epsilon) \sim \sum_0^\infty (-1)^n n!\epsilon^n$$

to denote the statement (88). (We will sometimes also denote the asymptotic expansion by the finite sum plus the remainder, or just the finite sum itself.) Since we know the asymptotic series in (88) diverges as $N \to \infty$, a natural question to ask is, how to decide where to truncate the series? Obviously for a given value of ϵ, as one takes more and more terms, the approximation must eventually get *worse*, since the terms diverge as $N \to \infty$. A given asymptotic expansion, for a given value of ϵ, will have an optimal truncation value $N(\epsilon)$. For this particular case, optimal truncation is for $N \sim 1/\epsilon$.

We now generalize this idea somewhat. The series does not need to be in powers of ϵ (though this simple form is frequently observed). We define an **asymptotic sequence** $\delta_n(\epsilon)$ to be a decreasing sequence of functions as $\epsilon \to 0$, such that

$$\left|\frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)}\right| \to 0 \quad \text{as } \epsilon \to 0.$$

Then, the notation

$$f(\epsilon) \sim \sum_{n=0}^{\infty} \delta_n(\epsilon) a_n \tag{89}$$

means

$$f(\epsilon) = \sum_{n=0}^{N} \delta_n(\epsilon) a_n + O(\delta_{N+1}(\epsilon)) \quad \text{as } \epsilon \to 0 \text{ for } N \text{ fixed.}$$

The right-hand side of (89) is then an asymptotic expansion of the function f, as $\epsilon \to 0$. The function f may also depend on a variable z of course, as well as the parameter ϵ, in which case the coefficients a_n in the above expansion will also be functions of z.

An asymptotic expansion may also be desired as a parameter k approaches some fixed nonzero value k_0, or even as $k \to \infty$. The case $k \to k_0$ is equivalent to the case $\epsilon \to 0$ discussed above if one identifies ϵ with $k - k_0$. For the case $k \to \infty$ we may identify ϵ with $1/k$ and again use the $\epsilon \to 0$ formalism.

Example 10.1 *Find an asymptotic expansion for*

$$f(k) = \int_0^\infty \frac{e^{-kt}}{1+t} dt$$

as $k \to \infty$, $k \in \mathbb{R}^+$.

Writing $\epsilon = 1/k$ we have

$$f = \int_0^\infty \frac{e^{-t/\epsilon}}{1+t}\, dt = \epsilon \int_0^\infty \frac{e^{-\tau}}{1+\epsilon\tau}\, d\tau.$$

The discussion above showed us how to obtain the asymptotic expansion of this integral as $\epsilon \to 0$; we have

$$f \sim \epsilon \sum_0^\infty (-1)^n n! \epsilon^n,$$

and thus

$$f \sim \sum_{n=1}^\infty \frac{(-1)^{n-1}(n-1)!}{k^n},$$

or equivalently,

$$f = \sum_{n=1}^N \frac{(-1)^{n-1}(n-1)!}{k^n} + R,$$

where $R = O(1/k^{N+1})$ as $k \to \infty$ at fixed N.

Example 10.2 *Find an asymptotic expansion for $I = \int_k^\infty \frac{e^{-t}}{t} dt$ as $k \to \infty$ ($k \in \mathbb{R}$).*

As before we may define

$$I_N = \int_k^\infty \frac{e^{-t}}{t^N} dt = \left[\frac{-e^{-t}}{t^N}\right]_k^\infty - N\int_k^\infty \frac{-e^{-t}}{t^{N+1}} dt = \frac{e^{-k}}{k^N} - NI_{N+1}.$$

Thus,

$$\begin{aligned} I_1 &= \frac{e^{-k}}{k} - 1.I_2 = \frac{e^{-k}}{k} - \left(\frac{e^{-k}}{k^2} - 2I_3\right) = \frac{e^{-k}}{k} - \frac{e^{-k}}{k^2} + 2\left(\frac{e^{-k}}{k^3} - 3I_4\right) \\ &\vdots \\ &= \frac{e^{-k}}{k} - \frac{e^{-k}}{k^2} + \frac{2!e^{-k}}{k^3} + \cdots + (-1)^{N-1}\frac{(N-1)!e^{-k}}{k^N} + (-1)^N N! I_{N+1}. \end{aligned}$$

This generates an asymptotic expansion for the integral as $k \to \infty$ provided $|N! I_{N+1}| \ll e^{-k}/k^N$ as $k \to \infty$, for fixed N. But clearly

$$|N! I_{N+!}| \leq \frac{N!}{k^{N+1}} \int_k^\infty e^{-t}\, dt = \frac{N! e^{-k}}{k^{N+1}} \ll \frac{e^{-k}}{k^N},$$

as $k \to \infty$, for fixed N. We again have the situation that the infinite series is divergent; but if we take only a finite number N of terms and let $k \to \infty$ then the remainder in the series goes to zero. Note that in this example the asymptotic series is not a simple power series.

10.2 Asymptotic expansions of complex functions

In the preceding discussion we have mostly supposed that the parameter in the asymptotic expansion is real, but asymptotic expansions can certainly be constructed for complex parameters or variables. If a function $f(z)$ is analytic on an infinite annulus $|z| > R$ then it has a convergent Laurent series expansion; and if in fact the function is bounded at infinity (= "analytic at infinity") also, then the Laurent series will be of the form

$$f(z) = \sum_{-\infty}^{0} c_n z^n = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots.$$

This (convergent) series is equivalent to an asymptotic expansion for the function (which happens to be convergent here). The corresponding asymptotic sequence is, of course, $(1/z^n)_{n=0}^\infty$. For a function f that is **not** analytic at infinity, there can be **no** convergent asymptotic expansion that is valid **uniformly in** z as $|z| \to \infty$. Instead we typically find that a function can have several different asymptotic expansions that are valid in different sectors of the complex plane. These need not be of the straightforward power-series type given above, but may be generalizations such as

$$f(z) \sim \Phi(z) \left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \right)$$

(cf example 10.2 above) or more complicated variants. It is a curious fact that two **different** functions can have the **same** asymptotic expansion in some region of the complex plane. Suppose that a function $f(z)$ has a convergent expansion on a sector $S_1 = \{z : -\pi/2 < \arg(z) < \pi/2\}$,

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \quad \text{valid on } -\pi/2 < \arg(z) < \pi/2. \tag{90}$$

Then the function $g(z) = f(z)+e^{-z}$ has an identical asymptotic power series expansion in S_1, because on that sector $-\pi/2 < \theta < \pi/2$ we have

$$|e^{-z}| = e^{-r\cos\theta} \ll |z^{-n}|, \quad \text{as } |z| \to \infty, \text{ for ANY } n > 0.$$

When we move outside S_1 however, the two functions will differ in their asymptotic expansions. If in fact the asymptotic expansion (90) is valid on an extended sector, say $S_2 = \{z : \pi/2 < \arg(z) < \pi\}$, then on the larger sector $S_1 \cup S_2$ f will have the same expansion as before, but the expansion of g on S_2 as $|z| \to \infty$ will now be dominated by the e^{-z}, which here is exponentially large:

$$g(z) \sim e^{-z} + \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad \text{as } |z| \to \infty.$$

It could be the case that on the remaining portion of the complex plane f itself has a different asymptotic expansion.

Note that in the above, the (convergent) Taylor series expansion of e^{-z} does NOT provide an asymptotic expansion, because its terms $T_n = (-1)^n z^n/n!$ do not satisfy the criterion of subsequent terms becoming asymptotically smaller as $|z| \to \infty$, for a fixed point N in the series (z^n is not an asymptotic sequence as $|z| \to \infty$). Therefore, if we were to truncate the series at some fixed value N, the truncated series would not provide a good approximation to the function for $|z|$ arbitrarily large.

This example with the function g is a very simple (and vague, at this stage) example of the **Stokes phenomenon**, in which the asymptotic expansion of a given function can change discontinuously across a line in complex space. We see that in the example above, the function e^{-z} possesses an essential singularity at infinity (consider the singular behavior of the function $e^{-1/z}$ at the origin), which is what gives rise to this behavior. Another simple example is given by the function $f(z) = \sinh(1/z)$, as $z \to 0$. This function has an essential singularity at the origin, as is easily seen by writing down its Laurent expansion about $z = 0$. The Laurent expansion, though convergent in any neighborhood of the origin, does not provide an asymptotic expansion for the function, because if we truncate it at a finite number of terms, we do not get a good approximation to the function – there is no "largest" term as $|z| \to 0$ that we can start our expansion with and order terms in decreasing size. Again, it is the exponential representation that gives the asymptotic expansion for the function, and different representations are appropriate in different regions of $\mathbb{C}$. The full representation of the function is $f(z) = (e^{1/z} - e^{-1/z})/2$; and

$$
\begin{aligned}
|\exp(1/z)| &= |\exp(r^{-1}(\cos\theta - i\sin\theta))| = e^{r^{-1}\cos\theta}, \\
|\exp(-1/z)| &= |\exp(-r^{-1}(\cos\theta - i\sin\theta))| = e^{-r^{-1}\cos\theta},
\end{aligned}
$$

so that in the sector $-\pi/2 < \theta < \pi/2$ (the right-half plane; $\theta = \arg(z)$) we have $\cos\theta > 0$, and

$$
f(z) \sim \frac{1}{2}e^{1/z} \quad \text{as } |z| \to 0
$$

is a valid asymptotic expansion for $\sinh(1/z)$, while in the sector $\pi/2 < \theta < 3\pi/2$ (the left-half plane) we have $\cos\theta < 0$, and

$$
f(z) \sim -\frac{1}{2}e^{-1/z} \quad \text{as } |z| \to 0
$$

is a valid asymptotic expansion for $\sinh(1/z)$.

10.3 The Stokes phenomenon

The above example shows that for a function analytic and single-valued in the neighborhood of infinity (or at another point), but not analytic **at** infinity, its asymptotic expansion can change discontinuously across certain lines in the complex plane, even though the function itself has no discontinuous behavior as $|z| \to \infty$. If a function is analytic at infinity then the asymptotic expansion varies continuously at infinity (as we traverse an arbitrarily large circular contour, for example). This discontinuous behavior that can arise in a function's asymptotic expansion is known as the **Stokes phenomenon**. We will not discuss it in detail here, but will illustrate it by means of a specific example, which demonstrates that even the functional form of the asymptotic expansion can be completely different in adjacent regions of the complex plane. Ablowitz & Fokas [1], from which the following is adapted, gives a fuller discussion of the phenomenon in §6.6.

Example 10.3 *Find the asymptotic behavior of*

$$
f(z) = \int_0^\infty \frac{e^{-zt}}{1+t^4}\,dt
$$

as $|z| \to \infty$, $z \in \mathbb{C}$.

Suppose first that $t \in \mathbb{R}^+$ in the integral, with the path of integration taken along the positive real axis. Then as $t \to \infty$, the integral converges provided $\Re(zt) > 0$, that is, provided $-\pi/2 < \theta < \pi/2$, where $\theta = \arg(z)$. In this case, for large $|z|$ the integrand is exponentially small except near $t = 0$. The dominant contribution will therefore come from this region; and we may expand the factor $(1+t^4)^{-1}$ in the integrand using the binomial expansion: $(1+t^4)^{-1} = 1 - t^4 + O(t^8)$. Thus, for $-\pi/2 < \arg(z) < \pi/2$ we have

$$f(z) \sim \int_0^\infty e^{-zt}\,dt - \int_0^\infty t^4 e^{-zt}\,dt + O\left(\int_0^\infty t^8 e^{-zt} dt\right).$$

Writing $I_n = \int_0^\infty t^n e^{-zt}\,dt$, we have

$$I_n = \left[\frac{e^{-zt}}{-z} t^n\right]_0^\infty + \frac{n}{z}\int_0^\infty e^{-zt} t^{n-1}\,dt = \frac{n}{z} I_{n-1} = \cdots = \frac{n!}{z^n} I_0 = \frac{n!}{z^{n+1}}.$$

Thus, for $-\pi/2 < \arg(z) < \pi/2$ (region 1),

$$f(z) \sim \frac{1}{z} - \frac{4!}{z^5} + O(z^{-9}), \quad \text{as } |z| \to \infty. \tag{91}$$

However, this expansion is certainly not valid for values of $\arg(z)$ outside this region. To find the correct far-field behavior in other regions, we consider a different integration path in t, allowing t complex.

Noting that the function $1/(1+t^4)$ decays as $|t| \to \infty$, and that Jordan's Lemma 9.7 applies to the integrand along (any portion of) a large semicircular contour in the right-half plane (the integral along any such circular arc will go to zero as the arc goes to infinity), we may form a closed contour by taking a large quarter-circle, with radii along the positive real t-axis and the negative imaginary t-axis. Then, applying the residue theorem to this contour, we have

$$\begin{aligned} f(z) &= \int_0^{-i\infty} \frac{e^{-zt}}{1+t^4}\,dt - 2\pi i \operatorname{Res}(e^{-zt}/(1+t^4); e^{-i\pi/4}) \\ &= \int_0^{-i\infty} \frac{e^{-zt}}{1+t^4}\,dt + \frac{i\pi}{2} e^{-ze^{-i\pi/4}} e^{3\pi i/4}. \end{aligned} \tag{92}$$

The integral in (92) converges provided $\Re(zt) > 0$, which now corresponds to a different range of $\arg(z)$, and a different region of the complex z-plane. With $z = re^{i\theta}$ and $t = se^{-i\pi/2}$, for r, $s \in \mathbb{R}^+$, we have

$$zt = rse^{i(\theta - \pi/2)}, \quad \Re(zt) = rs\cos(\theta - \pi/2),$$

and thus a convergent integral for $0 < \theta < \pi$ (region 2). So, we can now compute the far-field behavior of $f(z)$ for the range $-\pi/2 < \theta < \pi$ ((region 1) $\cup$ (region 2)). (Note that there is overlap between region 1 and region 2, so potentially two different asymptotic expansions for f on this overlap domain.)

To evaluate the far-field behavior in region 2 due to (92), note that in the contribution from the residue we have

$$e^{-ze^{-i\pi/4}} = e^{-r(\cos(\theta-\pi/4)+i\sin(\theta-\pi/4))},$$

which is exponentially small in $-\pi/4 < \theta < 3\pi/4$, but exponentially large in $3\pi/4 < \theta < \pi$.

The same approach as was used to obtain (91) can be applied to the integral (call it J) in (92), since again, on its region of convergence in the z-plane, the dominant contribution comes from small values of $|t|$. Writing $t = -i\eta$ in this integral, where $\eta \in \mathbb{R}^+$, we find

$$J = -i\int_0^\infty \frac{e^{iz\eta}}{1+\eta^4}\,d\eta.$$

Since the dominant contribution (for $|z| \gg 1$, $0 < \theta < \pi$) comes from $0 < \eta \ll 1$, we can again expand the factor $(1+\eta^4)^{-1}$ using the binomial, and writing $w = -iz$ we obtain (exactly as before)

$$\begin{aligned} J &= -i\left(\frac{1}{w} - \frac{4!}{w^5} + O(w^{-9})\right), \quad \text{as } |w| \to \infty, \\ &= \frac{1}{z} - \frac{4!}{z^5} + O(z^{-9}), \quad \text{as } |z| \to \infty. \end{aligned}$$

Thus, on the overlap region, (region 1) $\cap$ (region 2), $0 < \theta < \pi/2$, we have an exponentially small contribution from the residue, and using the approach for either region, we obtain the same result,

$$f(z) \sim \frac{1}{z} - \frac{4!}{z^5} + O(1/z^{-9}),$$

since the exponentially-small term $e^{-ze^{-i\pi/4}}$ is much smaller than any element of the asymptotic sequence in negative powers of z (so would never appear in the asymptotic expansion). So there is no discrepancy between the two approaches.

The above shows that, on the whole region $-\pi/2 < \theta < 3\pi/4$ (region 1, plus that part of region 2 where the exponential contribution in (92) is exponentially small), the appropriate asymptotic expansion is

$$f(z) \sim \frac{1}{z} - \frac{4!}{z^5} + O(1/z^{-9}), \quad \text{as } |z| \to \infty,$$

while on the remainder of region 2, $3\pi/4 < \theta < \pi$, the exponential term is far larger than any of the algebraic terms, so that the asymptotic expansion there is

$$f(z) \sim \frac{i\pi}{2} e^{-ze^{-i\pi/4}} e^{3\pi i/4} \quad \text{as } |z| \to \infty.$$

Hence the form of the asymptotic expansion of f changes discontinuously across the ray $\theta = 3\pi/4$ (a Stokes line for this example), as we move from one region to the next.

It remains, of course, to elucidate the behavior of f in the region 3, $-\pi < \theta < -\pi/2$.

References

[1] Ablowitz, M.J., Fokas, A.S. Complex variables: Introduction and Applications (2nd Edition) *Cambridge University Press* (2003).

[2] Abramowitz, M., Stegun, I.A. Handbook of mathematical functions. *Dover, New York* (1972).