Math 656 • Main Theorems in Complex Analysis • Victor Matveev

ANALYTICITY: CAUCHY-RIEMANN EQUATIONS (Theorem 2.1.1); review CRE in polar coordinates.

Proof: you should know at least how to prove that CRE are *necessary* for complex differentiability (very easy to show). The proof that they are *sufficient* is also easy if you recall the concept of differentiability of real functions in R^n .

INTEGRAL THEOREMS:

There are 6 theorems corresponding to 6 arrows in my hand-out. Only 4 of them are independent theorems, while the other two are redundant corollaries, including the important (yet redundant) Morera's Theorem (2.6.5).

Cauchy-Goursat Theorem is the main integral theorem, and can be formulated in several completely equivalent ways:

1. Integral of a function analytic in a simply-connected domain D is zero for any Jordan contour in D
2. If a function is analytic inside and on a Jordan contour C, its integral over C is zero.
3. Integral over a Jordan contour C is invariant with respect to smooth deformation of C that does not cross singularities of the integrand.
4. Integrals over open contours C_{AB} and C'_{AB} connecting A to B are equal if C_{AB} can be smoothly deformed into C'_{AB} without crossing singularities of the integrand.
5. Integral over a disjoint but piece-wise smooth contour (consisting of properly oriented external part plus internal pieces) enclosing no singularities of the integrand, is zero (proved by cross-cutting or the deformation theorem above).

Cauchy's Integral Formula (2.6.1) is the next important theorem; you should know its proof, its extended version and corollaries:

- Liouville Theorem 2.6.4: to prove it, apply the extended version of C.I.F. to the function derivative
- Max Modulus Principle 2.6.5: review the proof.
- The circle-average property of harmonic functions.

SERIES THEOREMS:

- Review of real analysis: Weierstrass M-test, ratio test, absolute convergence implies convergence, uniform convergence vs convergence, uniformly converging series of continuous functions is a continuous function.
- Corollary of Weierstrass M-test and ratio test: if a power series converges anywhere (at z^*), it converges in a disk $|z|<|z^*|$
- Term-wise differentiation and integration of converging power series: only convergence on the boundary of the disk may be altered by these operations.
- Taylor Series (3.2.2). Proof: from C.I.F. using properties of geometric series and term-wise integration
- Laurent Series (3.3.1). Proof: from C.I.F. using properties of geometric series and term-wise integration
- Taylor and Laurent series coefficients are unique! Hence, any convergent power series is a Taylor series.

Singularities: isolated (removable, pole, essential) vs non-isolated (branch point, cluster point, natural boundary, boundary jump / branch cut). Isolated singularities have a local Laurent series converging in a punctured disk around the singularity.

“Information Theory“ of analytic functions: The Identity Theorem

Zeros of an analytic function are isolated! Review definition of zero of order “m” to prove this.
Corollary: A function analytic in D is uniquely defined by its values on any set in D that has a limit point in D.
Note: C.I.F. and the Taylor series theorems also show how an analytic function is uniquely defined by a subset of its values

Related theorem: Monodromy theorem (3.5.3): analytic continuation from A to B along two paths C and C’ is unique if the only singularities between C and C’ are isolated.

Big Picard Theorem (without proof):

In any neighborhood of its essential singularity an analytic function accepts any value, with possibly one exception, infinitely many times (hence, the range of a function near essential singularity is *dense* on entire **C**)

RESIDUE THEOREM: simplest combination of the series theorems and the integral theorems. Review residue calculation.

OTHER THEOREMS:

Argument Principle (Proof is very direct); **Rouche’s Theorem** (Proof: argument principle for (f + g) / f); **Schwartz Reflection Principle**; **Open Mapping Theorem** (Proof: Max Modulus Principle combined with Roche’s Theorem); **Jordan’s Lemma**