Math 676 * October 20, 2011 * Midterm Exam

1. **(16pts)** For each system, carefully explain whether the solution is unique or not unique. If the solution is not unique, find and graph at least three distinct solutions, and make sure to double-check that all of them satisfy the same equation.

$$(a)\ \begin{cases} \dfrac{dx}{dt} = \sqrt{t}\cos^2 x \\ x(0) = 0 \end{cases} \quad (b)\ \begin{cases} \dfrac{dx}{dt} = \sqrt{x}\cos t \\ x(0) = 0 \end{cases} \quad (c)\ \begin{cases} \dfrac{dx}{dt} = \sqrt{x}\cos t \\ x(0) = 1 \end{cases}$$

2. **(16pts)** For each linear system, find the fundamental solution, find *all* invariant subspaces and their basis vectors, and classify the equilibrium. Are these two systems topologically conjugate? Are they diffeomorphic? Explain, without showing the mapping. *Please do not diagonalize*

$$(a)\ \begin{cases} x' = -x \\ y' = 3x - y \\ z' = -2y - z \end{cases} \quad (b)\ \begin{cases} x' = -x - 2y \\ y' = 2x - y \\ z' = -3z \end{cases}$$

3. **(16pts)** Find all equilibria and categorize their *linear* stability

$$\begin{cases} x' = x^2 - xy^2 \\ y' = 1 - e^{y-z} \\ z' = -2 + 2\sqrt{1 + x - y} \end{cases}$$

4. **(16pts)** Find the range of real constant λ for which the following mapping is a contraction mapping on the Banach function space $C^0[0,1]$. Give a definition of a Banach space.

$$T[f(x)] = x + f(x) + \lambda\int_0^1 \frac{f(y)\,dy}{1 + x + y}$$

5. **(24pts)** Answer the following questions for the non-linear system given below:
   i) Find all equilibria and classify their linear stability.
   ii) Is this system Hamiltonian?
   iii) Find the Lyapunov function $L(x,y)$ in the form given below. Is this a weak or a strong Lyapunov function? Does $L(x,y)$ help in determining the stability of both equilibria, or only one of them? Explain.
   iv) Classify the stability of all equilibria. Does the LaSalle's invariance principle apply to this case? If so, use it.
   v) Sketch the flow in phase space (at least near the origin). Note that the nullclines are very simple.

$$\begin{cases} x' = y \\ y' = -y - x - x^2 \end{cases} \quad L = \frac{1}{2}(x^2 + y^2) + cx^m, \text{ where } c \text{ is a real constant, and } m > 0 \text{ is an integer}$$

6. **(12pts)** Show that the following non-autonomous periodic linear system is unstable by calculating the determinant of the Monodromy matrix:

$$\begin{cases} x' = 4x\cos^2 t + y\cos t \\ y' = x\sin t - y\sin t \end{cases}$$

You may do the following problem instead of problem 6:

6'. **(12pts)** Which of the following statements are true for an *n* x *n* matrix *A*? Explain very briefly each answer

a) *A* has a one-dimensional invariant linear subspace in R^n

b) *A* has an n-dimensional invariant linear subspace in R^n

c) If n=3, *A* has a two-dimensional invariant linear subspace in R^3

d) If *A* is not invertible, then *A* must have a center linear subspace in R^n