Math 630-102  
Homework #12 (last one)  
Due date: April 26, 2007

Problem 1 (5.4.10). Decide on the stability or instability of the zero equilibrium for $dv/dt=w$, $dw/dt=v$. Is there a solution that decays to zero? Draw some arrows in the phase plane $(v, w)$ to explain your answer.

Problem 2 (Spectral decomposition)
Find the eigenvectors and the eigenvalues of the symmetric matrix $A=\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$, and verify the spectral decomposition of this matrix as a sum of two rank-one projection matrices, $A=\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$, where $q_1$ and $q_2$ are the two eigenvectors of $A$ normalized to unit length.

Problem 3 (Similarity transformation)
Consider the matrix $A=\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ from problem III of homework 10. The diagonalization $\Lambda = S^{-1} A S$ does not exist for this matrix, since it has only one linearly independent eigenvector. However, we can transform the matrix into an “almost” diagonal form, using these similarity transformations:

a) Place the eigenvector in the first column of a 2-by-2 matrix $M$. Take any vector orthogonal to the eigenvector, and place it in the second column of $M$. Show that the similarity transformation $B = M^{-1} A M$ yields a triangular matrix. What do the diagonal elements of $B$ say about the original matrix $A$?

b) Normalize the two columns of $M$ to unit length, and denote the resulting orthogonal matrix $Q$. Find the similar matrix $C = Q^T A Q$, and compare your result with matrix $B$ from part (a) (this particular similarity transformation appears in the Schur’s lemma).

c) Multiply the second column of matrix $M$ from part (a) by an arbitrary constant $c$, and find the new similar matrix $J=M^{-1} A M$. Find the value of $c$ so that the off-diagonal term of $J$ equals to one. This matrix $J$ is called the Jordan form.

Problem 4 (Jordan form)
Consider the difference equation $u_k = J^k u_{k-1}$, where $J=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ is a non-diagonalizable matrix in the so-called Jordan form. Multiply $J$ by itself a couple of times to figure out the general expression for $J^k$. Then, find the solution $u_k = J^k u_0$ for a general initial condition $u_0 = [c_1 \ c_2]^T$. 