Problem 1 (25 points)

The quadratic \( f = 3(x_1 + 2x_2)^2 + 4x_3^2 \) is positive. Find its matrix \( A \), factor it into \( LDL^T \), and connect the entries in \( D \) and \( L \) to the original \( f \).

Problem 2 (25 points)

Show from the eigenvalues that if \( A \) is positive definite, so are \( A^2 \) and \( A^{-1} \).

Problem 3 (25 points)

Decide whether the following matrices are positive definite, negative definite, semi-definite, or indefinite:

\[
A = \begin{bmatrix}
  1 & 2 & 3 \\
  2 & 5 & 4 \\
  3 & 4 & 9
\end{bmatrix}, \quad B = \begin{bmatrix}
  1 & 2 & 0 & 0 \\
  2 & 6 & -2 & 0 \\
  0 & -2 & 5 & -2 \\
  0 & 0 & -2 & 3
\end{bmatrix}, \quad C = -B, \quad D = A^{-1}.
\]

Is there a real solution to \( -x^2 - 5y^2 - 9z^2 - 4xy - 6xz - 8yz = 1 \)?

Problem 4 (25 points)

For the semi-definite matrices

\[
A = \begin{bmatrix}
  2 & -1 & -1 \\
 -1 & 2 & -1 \\
 -1 & -1 & 2
\end{bmatrix} \text{ (rank 2)} \quad \text{and} \quad B = \begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & 1 \\
  1 & 1 & 1
\end{bmatrix} \text{ (rank 1)}
\]

write \( x^T A x \) as a sum of two squares and \( x^T B x \) as one square.