1 Chapter 10.3: Ellipses

- An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points is a constant. The fixed points are called the foci of the ellipse
- The points of intersection of the ellipse with the line through the foci are called the vertices
- The line segment connecting the vertices is the major axis of the ellipse
- The midpoint of the major axis is the center of the ellipse
- The line segment that is perpendicular to the major axis at the center and with endpoints on the ellipse is the minor axis of the ellipse

Equation of an Ellipse
Let $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ and the $C = (0, 0)$. Consider a point on the ellipse given by $P = (x, y)$ and call the constant distance be $2a$. Then we have

$$\begin{aligned}
d(P, F_1) + d(P, F_2) &= 2a, \\
\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} &= 2a, \\
\sqrt{(x+c)^2 + y^2} &= 2a - \sqrt{(x-c)^2 + y^2}, \\
(x+c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2, \\
cx - a^2 &= -a\sqrt{(x-c)^2 + y^2}, \\
c^2x^2 - 2a^2cx + a^4 &= a^2((x-c)^2 + y^2), \\
(c^2 - a^2)x^2 + a^2y^2 &= a^2(a^2 - c^2)
\end{aligned}$$

Based on our definitions of a and c, it can be easily shown from a figure of an ellipse that $a > c$. The easiest way to see this is to draw a triangle where the vertices are F_1, F_2, P; then recall that the sum of two sides must always be greater than the third side.

Define $b^2 = a^2 - c^2 > 0$ and then divide the above equation by a^2b^2 to find

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } b^2 = a^2 - c^2 \tag{1}$$

This is the standard form equation of an ellipse with center $(0, 0)$ and foci $(\pm c, 0)$, where $b^2 = a^2 - c^2$. This is an example of a horizontal ellipse because the major axis is along or parallel to the x-axis.

In the above equation, if we instead place our foci at $(0, \pm c)$, then we'll find

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad \text{where } b^2 = a^2 - c^2 \tag{2}$$

an example of a vertical ellipse because the major axis is along or parallel to the y-axis.

The general form equation of an ellipse is found from expanding the standard form equation with a center not at that origin to find

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{3}$$

The graph of the above equation is an ellipse if and only if $AC > 0$: that is neither can be zero and both must have the same sign.

Example 10.3.1: Find an equation of an ellipse that has foci at $(2, -3)$ and $(2, 5)$ and has a major axis of length 10.
Solution: Because both foci lie on $x = 2$, the ellipse is a vertical ellipse. The center is the midpoint of the line joining the foci. Hence we can find

$$h = \frac{2+2}{2} = 2, \qquad k = \frac{-3+5}{2} = 1$$

The length of the major axis is $10 = 2a$ which implies that $a = 5$. We next notice that the distance from the foci to the center is $c = 4$. Hence we can find that $b^2 = a^2 - c^2 = 25 - 16 = 9 \implies b = 3$. Thus an equation for the ellipse is

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{25} = 1$$

Example 10.3.2: Find the center, vertices, and foci of the ellipse with equation $x^2 + 4y^2 - 6x + 8y - 29 = 0$.
Solution: We need to complete the square in both x and y:

$$x^2 + 4y^2 - 6x + 8y - 29 = 0,$$
$$(x^2 - 6x + \_\_\_) + (4y^2 + 8y + \_\_\_) = 29 + \_\_\_ + \_\_\_,$$
$$(x-3)^2 + 4(y+1)^2 = 42,$$
$$\frac{(x-3)^2}{42} + \frac{(y+1)^2}{21/2} = 1$$

Hence, we have that the center is $(3, -1)$, $a^2 = 42 \implies a = \sqrt{42}$, and $b^2 = \frac{21}{2} \implies b = \sqrt{\frac{21}{2}}$. We can also calculate $c = \sqrt{a^2 - b^2} = \frac{3\sqrt{14}}{2}$. Now we have all the information needed to say that the vertices are at $(3 \pm \sqrt{42}, 1)$, the minor vertices are at $(3, -1 \pm \sqrt{\frac{21}{2}})$, and that the foci are at $(3 \pm \frac{3\sqrt{14}}{2}, -1)$.

	Horizontal Ellipse	Vertical Ellipse
Standard Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
Center	(h, k)	(h, k)
Major Axis Along	$y = k$	$x = h$
Length of Major Axis	$2a$	$2a$
Minor Axis Along	$x = h$	$y = k$
Length of Minor Axis	$2b$	$2b$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Minor Vertices	$(h, k \pm b)$	$(h \pm b, k)$
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Symmetry	About $x = h$ and $y = k$	About $x = h$ and $y = k$

Remark: Note that as foci get closer to the center, the ellipse becomes more circular. When the foci are both at the same point (the center), the definition of an ellipse correlates to the definition of a circle. Thus, a circle can be thought of as a special case of an ellipse.