1 Chapter 4.1: Exponential Functions

A function of the form

$$f(x) = a^x, \quad a > 0, \quad a \neq 1 \tag{1}$$

is called an **exponential function** with **base a** and **exponent x**. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.

Rules of Exponents
Let a, b, x and y be real numbers with $a > 0$, $b > 0$. Then:

1. $a^x \cdot a^y = a^{x+y}$
2. $a^x / a^y = a^{x-y}$
3. $a^0 = 0$
4. $a^{-x} = 1/a^x$
5. $(a^x)^y = a^{xy}$
6. $(ab)^x = a^x b^x$
7. $(a/b)^x = a^x / b^x$

Example 4.1.1: Write in terms of a single irrational exponent $6^{\sqrt{27}} \cdot 6^{\sqrt{12}}$
Solution:

$$6^{\sqrt{27}} \cdot 6^{\sqrt{12}} = 6^{3\sqrt{3}} \cdot 6^{2\sqrt{3}} = 6^{5\sqrt{3}} = (6^5)^{\sqrt{3}} = 7776^{\sqrt{3}} \tag{2}$$

Graphing an Exponential: Let $y = f(x) = a^x,\ a > 0,\ a \neq 1$.

1. The domain is $(-\infty, \infty)$
2. The range is $(0, \infty)$
3. $f(0) = 1$ and $f(x+1) = a^{x+1} = a \cdot a^x = a \cdot f(x)$
   - This tells us that increasing x by 1 increases y by a.
4. For $a > 1$, the growth factor is a
   - f is an increasing function
   - as $x \to \infty$, we have $y \to \infty$
   - $x \to -\infty$, we have $y \to 0$
5. For $0 < a < 1$, the decay factor is a
   - f is a decreasing function
   - as $x \to \infty$, we have $y \to 0$
   - $x \to -\infty$, we have $y \to \infty$
6. Each exponential function is one-to-one
   - if $a^m = a^n$, then $m = n$
   - f has an inverse
7. The graph of f has no x-intercepts
8. The graph of f is smooth and continuous; it passes through the points $(-1, 1/a)$, $(0, 1)$, and $(1, a)$
9. The x-axis is a horizontal asymptote for the graph of every exponential function of the form $f(x) = a^x$
10. The graph of $y = a^{-x}$ is the reflection about the y-axis of the graph $y = a^x$

Transformations of $y = a^x$

Transformation	Appearance in Equation	Effect on (x, y)
Horizontal Shift	$y = a^{x \pm c}$	$(x \mp c, y)$
Horizontal Stretch/Compression	$y = a^{cx}$	$(x/c, y)$
Reflection about x-axis	$y = -a^x$	$(x, -y)$
Reflection about y-axis	$y = a^{-x}$	$(-x, y)$
Vertical Stretch/Compression	$y = ca^x$	(x, cy)
Vertical Shift	$y = a^x \pm c$	$(x, y \pm c)$

Table 1: This table gives all the transformations that can be used to plot exponential functions from the parent function $y = a^x$. The transformations can be applied in the order with which they are given.

Simple Interest

- A fee charged for borrowing a lender's money is the interest, denoted I
- The original, or initial, amount of money borrowed is the principal, denoted P
- The amount of interest computed only on the principal is simple interest

The simple interest, I, on a principal, P, at an annual interest rate of r (as a decimal) per year for t years is:

$$I = Prt \tag{3}$$

Compound Interest

- The amount of interest computed on the principal and accrued (previously earned) interest is compound interest

The compound interest, A, is computed using the following formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \tag{4}$$

where P is the principal, r is the annual interest rate, n is the number of times interest is compounded each year, and t is the number of years that have passed.

Continuous Compounding When interest is compounded continuously, the amount A after t years is given by

$$A = Pe^{rt} \tag{5}$$

where e is Euler's number.

- Note that Eq. (4) and Eq. (5) imply that $e = \lim_{n\to\infty}(1 + 1/n)^n$

Models for Exponential Growth and Decay Exponential growth and decay can be expressed as

$$A(t) = A_0 e^{kt} \tag{6}$$

where $A(t)$ is the amount at time t, A_0 is the initial amount, k is the relative rate of growth ($k > 0$) or decay ($k < 0$), and t is the time.

Example 4.1.2: In 2000 the population was 6.08 billion. Assume the annual rate of growth from 1990 onward is 1.5%. Using an exponential model, estimate the population at 2030 and 1990.
Solution: We start by noticing that $k = .015$, $A_0 = 6.08$, $t_1 = 30$, and $t_2 = -10$ where t_1 will be used to calculate the population at 2030 and t_2 will be used to calculate the population at 1990. Then we find:

$$A(t) = A_0 e^{kt} = 6.08\ e^{.015t}, \tag{7}$$

$$A(30) = 6.08\ e^{.015\cdot(30)} = 9.54 \text{ billion people}, \tag{8}$$

$$A(= 10) = 6.08\ e^{.015\cdot(-10)} = 5.23 \text{ billion people}. \tag{9}$$