1 Chapter 4.2: Logarithmic Functions

The Logarithmic Function
For $x > 0$, $a > 0$ and $a \neq 1$

$$y = \log_a(x) \quad \text{if and only if} \quad x = a^y \tag{1}$$

The function $f(x) = \log_a(x)$ is called the logarithmic function with base a. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.

Example 4.2.1: Write in logarithmic form:

$$4^3 = 64, \qquad \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

Solution: Notice that by Eq. (1), the base in these examples is 4 and 1/2 respectively. Hence we can rearrange these as

$$3 = \log_4 64, \qquad 4 = \log_{\frac{1}{2}}\left(\frac{1}{16}\right)$$

Basic Properties of Logs
For any base $a > 0$, with $a \neq 1$:

1. $\log_a(a) = 1$
2. $\log_a(1) = 0$
3. $\log_a(a^x) = x$ for any real number x
4. $a^{\log_a(x)} = x$ for any $x > 0$

Example 4.2.2: Evaluate:

$$9^{\log_9(6)} + \log_4(4^{-3}), \qquad \log_5(5^7) - 6^{\log_6(3)}$$

Solution: The two expressions can be evaluated by applying properties 3 and 4 to find:

$$9^{\log_9(6)} + \log_4(4^{-3}) = 6 + (-3) = 3, \qquad \log_5(5^7) - 6^{\log_6(3)} = 7 - 3 = 4$$

Inverting the Exponential
Swap x and y in the equation for an exponential and take the logarithm of both sides to find:

$$y = a^x \to x = a^y \implies \log_a(x) = \log_a(a^y) = y\log_a(a) = y \tag{2}$$

That is, the inverse of the exponential function is $y = \log_a(x)$, which is the logarithmic function.

Graphing a Logarithm
Let $y = f(x) = \log_a x$. Then

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$
2. The x-intercept is 1 and there is no y-intercept
3. The y-axis is the vertical asymptote
4. The function is one-to-one
5. If $a > 1$
   - The function is increasing
   - as $x \to \infty$, we have $y \to \infty$
   - $x \to 0^+$, we have $y \to -\infty$
6. If $0 < a < 1$

- The function is decreasing
- as $x \to \infty$, we have $y \to -\infty$
- $x \to 0^+$, we have $y \to \infty$

7. The graph of $y = \log_a x$ always contains the points $(1/a, -1)$, $(1, 0)$, and $(a, 1)$

Transformation	Appearance in Equation	Effect on (x, y)
Horizontal Shift	$y = \log_a (x \pm c)$	$(x \mp c, y)$
Horizontal Stretch/Compression	$y = \log_a (cx)$	$(x/c, y)$
Reflection about x-axis	$y = -\log_a (x)$	$(x, -y)$
Reflection about y-axis	$y = \log_a (-x)$	$(-x, y)$
Vertical Stretch/Compression	$y = c \log_a (x)$	(x, cy)
Vertical Shift	$y = \log_a (x) \pm c$	$(x, y \pm c)$

Table 1: This table gives all the transformations that can be used to plot exponential functions from the parent function $y = \log_a (x)$. The transformations can be applied in the order with which they are given.

Common and Natural Logarithms

- The logarithm with base 10 is called the common logarithm and is denoted by omitting the base:

$$\log_{10} (x) = \log (x)$$

- The logarithm with base e is called the natural logarithm and is denoted by:

$$\log_e (x) = \ln (x)$$

Note how the basic properties translate when $a = 10$ or $a = e$: **Continuous Compound Interest**

$\log_a (a) = 1$	$\log (10) = 1$	$\ln (e) = 1$
$\log_a (1) = 0$	$\log (1) = 0$	$\ln (1) = 0$
$\log_a (a^x) = x$	$\log (10^x) = x$	$\ln (e^x) = x$
$a^{\log_a (x)} = x$	$10^{\log (x)} = x$	$e^{\ln (x)} = x$

Recall the formula for continuous compound interest is given by

$$A = Pe^{rt} \tag{3}$$

where A is the compound interest, P is the principal, r is the annual interest rate, and t is the number of years. We can use the logarithmic function to rearrange this to solve for the variables in the exponent as follows:

$$A = Pe^{rt},$$

$$\frac{A}{P} = e^{rt},$$

$$\ln \left(\frac{A}{P} \right) = \ln \left(e^{rt} \right) = rt \tag{4}$$

The following example illustrates when this is useful.

Example 4.2.3: (a) How long will it take to double your money if it earns 6.5% compounded continuously?
(b) At what rate of return, compounded continuously, would your money double in 5 years?
Solution: (a) In this example, we are given $r = .065$, P, $A = 2P$. Hence our equation reads

$$\ln \left(\frac{2P}{P} \right) = .065t \implies t = \frac{\ln 2}{.065} \approx 10.66 \text{ years}$$

(b) In this example, we have $t = 5$, P, $A = 2P$. Hence our equation reads

$$\ln \left(\frac{2P}{P} \right) = 5r \implies r = \frac{\ln 2}{5} \approx .1386 = 13.86\ \%$$

Example 4.2.4: Find the domain of the following functions

$$f(x) = \frac{\ln(x-1)}{\ln(2-x)}, \qquad g(x) = \ln(x-3) + \ln(2-x)$$

Solution: The argument of the logarithmic function must be positive. Hence to find the domain of $f(x)$ we need to take the intersection of $x - 1 > 0$ and $2 - x > 0$. Thus we find that the domain of $f(x)$ is

$$(1, \infty) \cap (-\infty, 2) = (1, 2)$$

Similarly to find the domain of $g(x)$ we need to take the intersection of $x - 3 > 0$ and $2 - x > 0$. Thus we find that the domain of $g(x)$ is

$$(3, \infty) \cap (-\infty, 2) = \emptyset$$