1 Chapter 4.3: Rules of Logarithms

Rules of Logarithms
For M, N, a positive, real numbers with $a \neq 1$ and r a real number:

1. $\log_a(MN) = \log_a M + \log_a N$
2. $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
3. $\log_a(M^r) = r\log_a M$

Example 4.3.1: Given $\log_5 z = 3$ and $\log_5 y = 2$, find $\log_5(yz)$, $\log_5(125y^7)$, and $\log_5\left(\sqrt{\frac{z}{y}}\right)$.
Solution: We use the rules of logarithms to find

$$\log_5(yz) = \log_5(y) + \log_5(z) = 2 + 3 = 5$$

$$\log_5(125y^7) = \log_5(125) + \log_5(y^7) = 3 + 7\log_5(y) = 3 + 7(2) = 17$$

$$\log_5\left(\sqrt{\frac{z}{y}}\right) = \log_5\left(\sqrt{z}\right) - \log_5\left(\sqrt{y}\right) = \frac{1}{2}\log_5(z) - \frac{1}{2}\log_5(y) = \frac{3}{2} - 1 = \frac{1}{2}$$

Change-of-Base Formula
Let a, b, x be positive numbers with $a \neq 1, b \neq 1$. Then $\log_b x$ can be converted to a logarithm of a different base as follows

$$\log_b x = \frac{\log_a x}{\log_b x} = \frac{\log x}{\log b} = \frac{\ln x}{\ln b} \tag{1}$$

Example 4.3.2: Find an equation of the form $y = c + b\log x$ whose graph contains the points $(2, 3)$ and $(4, -5)$.
Solution: We plug the given points to find

$$3 = c + b\log 2, \qquad -5 = c + b\log 4 = c + 2b\log 2$$

We can now subtract the equations to find

$$8 = -b\log 2 \implies b = -\frac{8}{\log 2}$$

Substituting this into the other equation we find $c = 11$. Hence the equation we want is

$$y = 11 - \frac{8}{\log 2}\log x = 11 - 8\log_2 x$$

Example 4.3.3: In an experiment, 18 grams of radioactive element sodium-24 decayed to 6 grams in 24 hours. Find its half-life to the nearest hour.
Solution: The starting point is the exponential decay model given by

$$A = A_0 e^{kt} \to 6 = 18e^{24k} \implies k = -\frac{\ln 3}{24}$$

To find the half-life, we need to solve for t such that $A = A_0/2$:

$$\frac{A_0}{2} = A_0 e^{kt} \implies \frac{1}{2} = e^{kt} \implies \ln\left(\frac{1}{2}\right) = kt \implies t = -\frac{\ln 2}{k} = \frac{24\ln 2}{\ln 3} \approx 15 \text{ hours}$$

Example 4.3.4: Evaluate $5^{2\log_5 3 + \log_5 2}$ and $\log 4 + 2\log 5$
Solution:

$$5^{2\log_5 3+\log_5 2} = 5^{2\log_5 3} \cdot 5^{\log_5 2} = 5^{\log_5 9} \cdot 5^{\log_5 2} = (9)(2) = 18,$$
$$\log 4 + 2\log 5 = \log 4 + \log 25 = \log 100 = 2$$